

Multiplicity free quasi-Hamiltonian manifolds

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ABSTRACT. A (quasi-)Hamiltonian manifold is called multiplicity free if all of its symplectic reductions are 0-dimensional. In this paper, we classify multiplicity free Hamiltonian actions for (twisted) loop groups or, equivalently, multiplicity free (twisted) quasi-Hamiltonian manifolds for simply connected compact Lie groups. As a result we recover old and find new examples of these structures.

1. Introduction

In [AMM98], Alekseev-Malkin-Meinrenken introduced the concept of a *group valued moment map* and showed that it is essentially equivalent to the concept of Hamiltonian loop groups actions. The advantage of the former is that all objects are finite dimensional. As opposed to the ordinary moment map which takes values in the coadjoint representation of the acting Lie group K , a group valued moment map takes values in K itself (which, in this paper, is usually assumed to be compact and simply connected). A manifold equipped with a group valued moment map is called *quasi-Hamiltonian*.

Like Hamiltonian manifolds, quasi-Hamiltonian manifolds also have a notion of symplectic reduction and the dimension of the symplectic reductions of a quasi-Hamiltonian manifold serves as a measure for its size. At the bottom of this hierarchy lie the so called *multiplicity free* (quasi-Hamiltonian) manifolds, i.e., those manifolds having 0-dimensional reductions. The purpose of the present paper is to classify multiplicity free quasi-Hamiltonian manifolds under two technical conditions: the group K should be simply connected and the manifold should be convex. The first condition is essential while the second is rather mild and holds automatically for compact manifolds.

Actually, it turned out to be more natural to consider also *twisted* quasi-Hamiltonian manifolds. This means that K is equipped with an automorphism τ such that the moment map to K is equivariant with respect to twisted conjugation $g \mapsto kg\tau(k)^{-1}$. In analogy with [AMM98], twisted quasi-Hamiltonian manifolds correspond to Hamiltonian actions of twisted loop groups. Here a twisted loop is a map $g : \mathbb{R} \rightarrow K$ satisfying the periodicity condition $g(t+1) = \tau(g(t))$.

The classification is in terms of two data. The first encodes the image of the moment map $m : M \rightarrow K$. For this recall that the set of (twisted) conjugacy classes of K is in bijection with an alcove \mathcal{A} for a certain affine root system (see, e.g., [MW04]). Therefore, the image of m is determined by a subset $\mathcal{P}_M \subseteq \mathcal{A}$ which turns out to be a convex polyhedron, if M is compact and which is, in general, locally polyhedral. The second

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datum is a lattice Λ_M which encodes the principal isotropy group of K on M . These two objects \mathcal{P}_M and Λ_M satisfy certain compatibility conditions called sphericity (see below for more) and our main result, Theorem 6.7, says that convex multiplicity free quasi-Hamiltonian manifolds are classified by spherical pairs (\mathcal{P}, Λ) . Observe that the uniqueness part is a quasi-Hamiltonian generalization of the Delzant conjecture [Del90] (which is proved in [Kno11]).

The main application of our classification is the construction of quasi-Hamiltonian manifolds. This is much harder than in the Hamiltonian setting since given a subgroup $H \subseteq K$ it is, in general, not possible to restrict a K -valued moment map to an H -valued one. So our approach is to construct spherical pairs (\mathcal{P}, Λ) which then lead to quasi-Hamiltonian manifolds.

Using this technique, we are able to recover most examples which have been previously constructed “by hand”: The double of a group by Alekseev-Malkin-Meinrenken, [AMM98], the spinning 4-sphere by Alekseev-Meinrenken-Woodward, [AMW02], its generalization, the spinning $2n$ -sphere by Hurtubise-Jeffreys-Sjamaar, [HJS06], and the quaternionic projective space due to Eshmatov, [Esh09].

Beyond that, we show that more generally the quaternionic Grassmannians $\mathrm{Gr}_k(\mathbb{H}^{n+1})$ carry a quasi-Hamiltonian $Sp(2n)$ -structure. We also find compact quasi-Hamiltonian manifolds for the groups $SU(n)$ and $Sp(2n)$, respectively, for which the moment map is surjective, something which has no analogue for Hamiltonian manifolds. On the side we observe that the product of any two symmetric spaces for the same group with diagonal action (we call them *disymmetric*) carries a multiplicity free quasi-Hamiltonian structure. This explains many of their nice invariant theoretic properties.

Some words on the sphericity condition. Recall that a (complex algebraic) variety X with an action of a connected reductive group G is called *spherical* if a Borel subgroup of G has a dense open orbit in X . When X is affine this has a purely representation theoretic interpretation: X is spherical if and only if its coordinate ring $\mathbb{C}[X]$ is a multiplicity free G -module. In that case, $\mathbb{C}[X]$ is, as a G -module, uniquely determined by the set Λ_X^+ of highest weights occurring in it. If X is additionally smooth then it is even uniquely determined by Λ_X^+ (Losev [Los09]).

Now it follows from work of Brion [Bri87] and Sjamaar [Sja98] that a multiplicity free manifold M is locally modeled after a smooth affine spherical variety. This means the following: For any $x \in \mathcal{P}_M$ let $L \subseteq K$ be the (twisted) centralizer of $\exp(x)$ in K . Then an open K -invariant neighborhood of $m^{-1}(x)$ in M is isomorphic to an open subset of a “model space” of the form $K \times^L X$ where X is a smooth affine spherical $L_{\mathbb{C}}$ -variety. Strictly speaking, Brion and Sjamaar proved this only for ordinary Hamiltonian manifolds but using techniques from [AMM98] it readily generalizes to the twisted quasi-Hamiltonian setting.

This has the following consequence for the pair $(\mathcal{P}, \Lambda) := (\mathcal{P}_M, \Lambda_M)$. Let C_X be the convex cone and Λ_X the abelian group Λ_X generated by Λ_M^+ . Then

$$C_x \mathcal{P} = C_X \text{ and } \Lambda = \Lambda_X \tag{1.1}$$

where $C_x \mathcal{P} = \mathbb{R}_{\geq 0}(\mathcal{P} - x)$ is the tangent cone of \mathcal{P} in x . Conversely, we call a pair (\mathcal{P}, Λ) spherical if for every $x \in \mathcal{P}$ there is a smooth affine spherical $L_{\mathbb{C}}$ -variety X such that (1.1) holds.

Admittedly, the sphericity condition is not very explicit since it involves finding an appropriate variety X for every point $x \in \mathcal{P}$. This task is simplified by two facts: First of all, it suffices to check it for representatives of the minimal faces of \mathcal{P} . So if \mathcal{P} is a polyhedron then it is enough to check the vertices. Secondly, Van Steirteghem and the author have essentially classified all smooth affine spherical varieties in [KVS06]. A description of their weight monoids will appear in joint work with Pezzini and Van Steirteghem [KPVS]. These works make it possible to test sphericity in any given vertex. Conversely, one can use the classification to look for local models such that the tangent cones and the lattice (according to (1.1)) paste to a global spherical pair. See section 11 for examples on how this strategy works.

This paper is, to a certain extent, a sequel of [Kno11] where analogous results were proved in the Hamiltonian setting. Nevertheless, I have tried to make the present paper self-contained enough such that at least the main results should be understandable without consulting [Kno11] or even [AMM98]. On the other hand, for some of the main arguments, especially the local structure of multiplicity free Hamiltonian manifolds and their automorphism groups, we refer to [Kno11]. The cohomology computations in section 10 are quite a bit more involved for quasi-Hamiltonian manifolds than those of [Kno11].

We include systematically the twisted case and most multiplicity free examples in this setting seem to be new. After completion of a first version of this paper, the author became aware of Meinrenken's manuscript [Mei15] which explicitly studies twisted quasi-Hamiltonian manifolds. Therefore, it has some overlap with sections 2, 4, and 5.

Finally some advice for reading the paper: I gathered all examples in the final section 11 but most of them can be understood much sooner.

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2. From Hamiltonian loop group spaces to quasi-Hamiltonian manifolds

In this section, we give a brief introduction to Hamiltonian actions of twisted loop groups. The untwisted case has been worked out by Alekseev, Malkin and Meinrenken in their nice paper [AMM98]. For a short survey see also §1.4 of [GS05]. Therefore, the main purpose of this and the following section is to make precise where to put the twist τ in.

In the whole paper, let K be a compact connected Lie group with Lie algebra \mathfrak{k} . Let τ be a fixed “twist” of K which simply means a continuous automorphism of K . The action of τ on $k \in K$ is going to be denoted by ${}^\tau k$. The induced automorphism of \mathfrak{k} will also be denoted by τ .

We also fix a K - and τ -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{k} . For semisimple K one could take the Killing form but it will be useful to also consider differently scaled scalar products (see, e.g., the final remark of example 2 in section 11).

The *twisted loop group* $\mathcal{L}_\tau(K)$ is the set of smooth maps $g : \mathbb{R} \rightarrow K$ which are subject to the condition

$$g(t+1) = {}^\tau g(t) \text{ for all } t \in \mathbb{R}. \quad (2.1)$$

It is a group under pointwise multiplication. Clearly, if $\tau = \text{id}_K$ then $\mathcal{L}_\tau(K)$ is just the group of smooth loops $g : S^1 = \mathbb{R}/\mathbb{Z} \rightarrow K$.

The space $\mathcal{L}_\tau(\mathfrak{k})$ of smooth maps $\xi : \mathbb{R} \rightarrow \mathfrak{k}$ with $\xi(t+1) = {}^\tau\xi(t)$ is the Lie algebra of $\mathcal{L}_\tau(K)$. The invariant scalar product on \mathfrak{k} induces a scalar product on $\mathcal{L}_\tau(\mathfrak{k})$ by

$$\langle \xi, \eta \rangle_{\mathcal{L}} := \int_0^1 \langle \xi(t), \eta(t) \rangle dt. \quad (2.2)$$

Observe that the integrand is periodic of period 1, hence integration over any interval of length 1 yields the same result. It also follows that

$$c(\xi, \eta) := \langle \xi', \eta \rangle_{\mathcal{L}} = -\langle \xi, \eta' \rangle_{\mathcal{L}}. \quad (2.3)$$

(where ξ', η' are the derivatives with respect to t) is a 2-cocycle and therefore defines a central extension $\widehat{\mathcal{L}}_\tau(\mathfrak{k}) := \mathcal{L}_\tau(\mathfrak{k}) \oplus \mathbb{R}\kappa$ of $\mathcal{L}_\tau(\mathfrak{k})$ by

$$[\xi + s\kappa, \eta + t\kappa] := [\xi, \eta] + c(\xi, \eta)\kappa. \quad (2.4)$$

Dually, the space $\widehat{\mathcal{L}}_\tau(\mathfrak{k}^*) := \mathcal{L}_\tau(\mathfrak{k}) \oplus \mathbb{R}E$ is in duality with $\widehat{\mathcal{L}}_\tau(\mathfrak{k})$ by

$$\langle A + sE, \xi + t\kappa \rangle_{\mathcal{L}} := \langle A, \xi \rangle_{\mathcal{L}} + st. \quad (2.5)$$

The dual (coadjoint) action of $\mathcal{L}_\tau(\mathfrak{k})$ on $\widehat{\mathcal{L}}_\tau(\mathfrak{k}^*)$ is then

$$\xi \cdot (A + sE) = [\xi, A] - t\xi'. \quad (2.6)$$

Therefore the action of $g(t) \in \mathcal{L}_\tau(K)$ on $A(t) + sE \in \widehat{\mathcal{L}}_\tau(\mathfrak{k}^*)$ is

$$g \cdot (A + sE) = (\text{Ad}(g)A - sg'g^{-1}) + sE. \quad (2.7)$$

Since the so-called *level* $s = \langle \cdot, \kappa \rangle_{\mathcal{L}}$ is invariant under this action, the group $\mathcal{L}_\tau(K)$ acts on the level-1-set $\mathcal{L}_\tau^1(\mathfrak{k}) := \mathcal{L}_\tau(\mathfrak{k}) + E$. Under the identification $\mathcal{L}_\tau(\mathfrak{k}) \cong \mathcal{L}_\tau^1(\mathfrak{k}) : A \mapsto A + E$ this action becomes

$$g \cdot A = \text{Ad}(g)A - g'g^{-1}. \quad (2.8)$$

2.1. Definition. A *Hamiltonian $\mathcal{L}_\tau(K)$ -space* (of level 1) is a Fréchet manifold X equipped with an $\mathcal{L}_\tau(K)$ -action, a 2-form σ , and map $\mu : X \rightarrow \mathcal{L}_\tau^1(\mathfrak{k})$ such that

- a) The map μ is smooth and equivariant with respect to the action (2.8).
- b) The 2-form σ is $\mathcal{L}_\tau(K)$ -invariant, closed and non-degenerate
- c) $\sigma(\xi x, \eta) = \langle \xi, \mu_*\eta \rangle_{\mathcal{L}}$ for all $\xi \in \mathcal{L}_\tau(\mathfrak{k})$ and $\eta \in T_x X$

The Hamiltonian space X is of *finite type* if μ is locally proper, i.e., every $x \in X$ has a (closed) neighborhood U such that $\mu|_U : U \rightarrow \mathcal{L}_\tau^1(\mathfrak{k})$ is proper.

Remarks. a) Because of its importance for geometry, it is customary to only consider a level of 1. Clearly, by rescaling this encompasses also actions of any positive but constant level. As a matter of fact, it appears that the whole theory of this paper should generalize to actions of non-constant level as long as it stays positive, i.e., to Hamiltonian $\widehat{\mathcal{L}}_\tau(K)$ -spaces with $\langle \mu(x), \kappa \rangle_{\mathcal{L}} > 0$ for all $x \in X$. Clearly, in such generality the center of $\widehat{\mathcal{L}}_\tau(K)$ will act non-trivially on X .

b) The $\mathcal{L}_\tau(\mathfrak{k})$ -equivariance of m is equivalent to the formula

$$\sigma(\xi x, \eta x) = \langle [\xi, \eta], \mu(x) \rangle_{\mathcal{L}} + c(\xi, \eta) \text{ for all } x \in X \text{ and } \xi, \eta \in \mathcal{L}_\tau(\mathfrak{k}). \quad (2.9)$$

Before we go on, we recall (and twist) the basic facts of the coadjoint loop group action. The key is the observation that any $A \in \mathcal{L}_\tau^1(\mathfrak{k})$ defines a connection on the trivial K -bundle $p : K \times \mathbb{R} \rightarrow \mathbb{R}$ (with K acting on the left). Then the gauge group $\mathcal{L}_\tau(L)$ acts on the right of $K \times \mathbb{R}$. The action on connections is given by equation (2.8). More precisely,

a horizontal section of the connection corresponding to A is a solution $z : \mathbb{R} \rightarrow K$ of the ordinary differential equation

$$z'(t) = z(t)A(t). \quad (2.10)$$

Any other solution is of the form $kz(t)$ with $k \in K$. In particular,

$$h_t(A) := z(0)^{-1}z(t) \quad (2.11)$$

depends only on A and not on the choice of z . The value at $t = 1$

$$h(A) := h_1(A) = z(0)^{-1}z(1) \in K \quad (2.12)$$

is called the *holonomy of A* . It is easily checked that if $g(t) \in \mathcal{L}_\tau(K)$ then $\tilde{z}(t) := z(t)g(t)^{-1}$ is a solution of (2.10) for $\tilde{A} := g \cdot A$ which implies

$$h_t(g \cdot A) = \tilde{z}(0)^{-1}\tilde{z}(t) = g(0)z(0)^{-1}z(t)g(t)^{-1} = g(0)h_t(A)g(t)^{-1}. \quad (2.13)$$

In particular, if we put $t = 1$ and observe that $g(1) = {}^\tau g(0)$ we get

$$h(g \cdot A) = g(0)h(A){}^\tau g(0)^{-1}. \quad (2.14)$$

The evaluation map $g \mapsto g(0)$ induces the short exact sequence

$$1 \mapsto \Omega_\tau(K) \rightarrow \mathcal{L}_\tau(K) \xrightarrow{\rightarrow} 1. \quad (2.15)$$

where $\Omega_\tau(K)$ is the group of based (twisted) loops, i.e., with $g(0) = 1$. Observe that, unlike in the untwisted case, the evaluation homomorphism does not have a canonical section. Then (2.14) says that h is $\mathcal{L}_\tau(K)$ -equivariant where K acts on itself by the twisted action

$$g \cdot_\tau k := gk{}^\tau g^{-1}. \quad (2.16)$$

To distinguish the twisted action from the adjoint action we are going to write $K\tau$ for K when we mean the former. This makes sense since, purely formally, we have

$$\text{Ad}(g)(k\tau) = gk\tau g^{-1} = gk(\tau g^{-1}\tau^{-1})\tau = (gk{}^\tau g^{-1})\tau. \quad (2.17)$$

Of course, this calculation can be made rigorous by considering $K\tau$ as a subset of the group $\mathbb{Z}\tau \ltimes K$.

The following fact is fundamental:

2.2. Lemma. *The map $h : \mathcal{L}_\tau^1(\mathfrak{k}) \rightarrow K\tau$ is an $\mathcal{L}_\tau(K)$ -equivariant principal fiber bundle for $\Omega_\tau(K)$.*

Proof. Consider a solution z of (2.10). Applying τ yields that both $z(t+1)$ and ${}^\tau z(t)$ are horizontal sections for ${}^\tau A$ which implies

$$z(t+1) = k{}^\tau z(t) \text{ where } k := z(0)h(A){}^\tau z(0)^{-1}. \quad (2.18)$$

Conversely, for every fixed element $k_0 \in K$ there is clearly a smooth map $z_0 : \mathbb{R} \rightarrow K$ with $z_0(t+1) = k{}^\tau z_0(t)$ and $z_0(0) = 1$. Then $A_0 := z_0(t)^{-1}z_0'(t)$ is an element of $\mathcal{L}_\tau^1(\mathfrak{k})$ with $h(A_0) = k_0$. This shows that h is surjective.

Now let $A_1 \in \mathcal{L}_\tau^1(\mathfrak{k})$ be a second element with $h(A_1) = k_0$. Let $z_1(t)$ be the corresponding horizontal section with $z_1(0) = 1$. Then (2.18) implies that $g(t) := z_1(t)^{-1}z_0(t)$ is an element of $\Omega_\tau(K)$. Moreover, an easy calculation shows $g \cdot A_1 = A_2$. Hence $\Omega_\tau(K)$ acts transitively on the fiber $h^{-1}(k_0)$. On the other hand, let $g(t) \in \Omega_\tau(K)$ with $g \cdot A_0 = A_0$. Then $\tilde{z} := z_0 g^{-1}$ is also a horizontal section for A with $\tilde{z}(0) = 1$. Hence, $\tilde{z} = z_0$ and therefore $g \equiv 1$ which means that the action of $\Omega_\tau(K)$ is free.

Finally, the bundle h is locally free since z_0 can be chosen to depend smoothly on k_0 in a small open subset. \square

We return to a Hamiltonian space X with moment map $\mu : X \rightarrow \mathcal{L}_\tau^1(\mathfrak{k})$. Since $\Omega_\tau(K)$ acts freely on the target, its action on X is free, as well. Let $\tilde{h} : X \rightarrow M := X/\Omega_\tau(K)$ be the quotient and $m := \mu/\Omega_\tau(\mathfrak{k}) : M \rightarrow K$. Then clearly, the following square is Cartesian:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathcal{L}_\tau^1(\mathfrak{k}) \\ \downarrow \tilde{h} & & \downarrow h \\ M & \xrightarrow{m} & K\tau \end{array} \quad (2.19)$$

Hence X and μ can be reconstructed from M and m . To also get the 2-form σ , Alekseev-Malkin-Meinrenken introduce in [AMM98] additional structure on M .

For this, let θ and $\bar{\theta}$ be the two canonical \mathfrak{k} -valued 1-forms on K , defined by $\theta(k\xi) = \xi = \bar{\theta}(\xi k)$. These are combined to get another \mathfrak{k} -valued 1-form

$$\Theta_\tau := \frac{1}{2}(\bar{\theta} + \tau^{-1}\theta) \quad (2.20)$$

with τ acting on the target \mathfrak{k} . Thus $\Theta_\tau(k\xi) = \frac{1}{2}(\text{Ad}(k)\xi + \tau^{-1}\xi)$.

Moreover, the scalar product on \mathfrak{k} induces the canonical biinvariant closed 3-form on K

$$\chi := \frac{1}{12}\langle \theta, [\theta, \theta] \rangle = \frac{1}{12}\langle \bar{\theta}, [\bar{\theta}, \bar{\theta}] \rangle. \quad (2.21)$$

2.3. Definition. A *quasi-Hamiltonian $K\tau$ -manifold* is a smooth manifold M equipped with a K -action, a 2-form ω , and a smooth map $m : M \rightarrow K\tau$, called the *(group valued) moment map*, having the following properties:

- a) m is K -equivariant.
- b) The form ω is K -invariant and satisfies $d\omega = -m^*\chi$.
- c) $\omega(\xi x, \eta) = \langle \xi, m^*\Theta_\tau(\eta) \rangle$ for all $\xi \in \mathfrak{k}$ and $\eta \in T_x M$.
- d) $\ker \omega_x = \{\xi x \in T_x M \mid \xi \in \mathfrak{k} \text{ with } m^{(x)\tau}\xi + \xi = 0\}$.

To get the connection with X , consider the family of \mathfrak{k} -valued 1-forms $\Xi_t := h_t^*\bar{\theta}$ on $\mathcal{L}_\tau^1(\mathfrak{k})$ and the 2-form

$$\varpi := \frac{1}{2} \int_0^1 \langle \Xi_t, \frac{d\Xi_t}{dt} \rangle dt. \quad (2.22)$$

One of the main results of [AMM98] is:

2.4. Theorem. *Let (X, σ, μ) be a Hamiltonian $\mathcal{L}_\tau(K)$ -space which is of finite type.*

- a) *Then there is a unique 2-form ω on $M = X/\Omega_\tau(K)$ such that $\sigma + \mu^*\varpi = \tilde{h}^*\omega$.*
- b) *The triple (M, ω, m) is a quasi-Hamiltonian $K\tau$ -manifold.*

Moreover, the functor $(X, \sigma, \mu) \mapsto (M, \omega, m)$ is an equivalence of categories between Hamiltonian $\mathcal{L}_\tau(K)$ -spaces and quasi-Hamiltonian $K\tau$ -manifolds (both with morphisms=isomorphisms).

Proof. The quasi-inverse functor is the fiber product $X = M \times_K \mathcal{L}_\tau^1(\mathfrak{k})$. In the untwisted case, this has been proved in [AMM98]. The main point is the derivation formula for $\iota_{v_\xi} \varpi$ in [AMM98, Appendix A] which now yields

$$\begin{aligned} \iota_{v_\xi} \varpi &= -d_A \langle A, \xi \rangle_{\mathcal{L}} + \frac{1}{2} \langle h^* \bar{\theta}, \xi(0) \rangle + \frac{1}{2} \langle h^* \theta, \xi(1) \rangle = \\ &= -d_A \langle A, \xi \rangle_{\mathcal{L}} + \langle h^* \Theta_\tau, \xi(0) \rangle. \end{aligned} \quad (2.23)$$

Here, v_ξ is the vector field on $\mathcal{L}_\tau^1(\mathfrak{k})$ which is induced by the action given by (2.8) of $\xi \in \mathcal{L}_\tau(\mathfrak{k})$. \square

Let's call a Hamiltonian $\mathcal{L}_\tau(K)$ -space X *complete* if its moment map $\mu : X \rightarrow \mathcal{L}_\tau^1(\mathfrak{k})$ is proper. This leads to the following observation:

2.5. Corollary. *Let M be the quasi-Hamiltonian $K\tau$ -manifold associated to the Hamiltonian $\mathcal{L}_\tau(K)$ -space X . Then X is complete if and only if M is compact.*

Now we are able to introduce the main objects of the present paper. To motivate it, consider one of the most important operations for Hamiltonian manifolds namely *symplectic reduction*. Assume that $A \in \mathcal{L}_\tau^1(\mathfrak{k})$ is in the image of the moment map. Then

$$X_A := \mu^{-1}(A) / \mathcal{L}_\tau(K)_A = \mu^{-1}(\mathcal{L}_\tau(K) \cdot A) / \mathcal{L}_\tau(K) \quad (2.24)$$

is called the symplectic reduction of X at A . Clearly, it depends only on the coadjoint orbit defined by A . Another way to describe it is to consider the image $a := h(A) \in K$. Then $X_A = M_a$ where

$$M_a := m^{-1}(a) / K_a = m^{-1}(Ka) / K. \quad (2.25)$$

It is known that for general a , the space M_a is a symplectic manifold whose (even) dimension is independent of a . So we define the *complexity* of X or M as

$$c(X) = \frac{1}{2} \dim X_A = \frac{1}{2} \dim M_a = c(M). \quad (2.26)$$

The most basic case is that of complexity zero, i.e., where all symplectic reductions are discrete.

2.6. Definition. A Hamiltonian $\mathcal{L}_\tau(K)$ -space or a quasi-Hamiltonian $K\tau$ -manifold of complexity zero is called *multiplicity free*.

Clearly, if M corresponds to X then one is multiplicity free if and only if the other is. The purpose of this paper is to classify complete multiplicity free Hamiltonian $\mathcal{L}_\tau(K)$ -spaces and compact multiplicity free quasi-Hamiltonian $K\tau$ -manifolds. Since both problems are equivalent, we will investigate only quasi-Hamiltonian manifolds from now on.

3. Affine root systems

Next, we need to recall some facts about twisted conjugacy classes. For this and also to state the classification result, we need to set up notation about affine root systems. Here we are following mostly Macdonald's [Mac72] and [Mac03].

Let $\bar{\mathfrak{a}}$ be an Euclidean vector space, i.e., a finite dimensional \mathbb{R} -vector space equipped with a positive definite scalar product $\langle \cdot, \cdot \rangle$ and let \mathfrak{a} be an affine space for $\bar{\mathfrak{a}}$, i.e., a set with a free and transitive $\bar{\mathfrak{a}}$ -action

$$\mathfrak{a} \times \bar{\mathfrak{a}} \rightarrow \mathfrak{a} : (x, t) \mapsto x + t. \quad (3.1)$$

The set of affine linear functions on \mathfrak{a} is denoted by $A(\mathfrak{a})$. It is an extension of the dual space \mathfrak{a}^* by the constant functions $\mathbb{R}1$. The gradient of $\alpha \in A(\mathfrak{a})$ is denoted by $\bar{\alpha} \in \bar{\mathfrak{a}}$. It is characterized by

$$\alpha(x+t) = \alpha(x) + \langle \bar{\alpha}, t \rangle, \quad x \in \mathfrak{a}, t \in \bar{\mathfrak{a}} \quad (3.2)$$

Similarly, let $M(\mathfrak{a})$ be the group of isometries of \mathfrak{a} (a.k.a. motions). It is an extension of the orthogonal group $O(\bar{\mathfrak{a}})$ by the group of translations $\bar{\mathfrak{a}}$. More precisely, the projection

$$M(\mathfrak{a}) \rightarrow O(\bar{\mathfrak{a}}) : w \mapsto \bar{w} \quad (3.3)$$

is characterized by the property

$$w(x+t) = w(x) + \bar{w}(t), \quad x \in \mathfrak{a}, t \in \bar{\mathfrak{a}}. \quad (3.4)$$

For a subgroup W of $M(\mathfrak{a})$ let \bar{W} be its image in $O(\bar{\mathfrak{a}})$.

A reflection is a motion $s \in M(\mathfrak{a})$ whose fixed point set is an affine hyperplane. If $\alpha \in A(\mathfrak{a})$ is a non-constant affine linear function with zero-set $H_\alpha := \alpha^{-1}(0)$ then

$$s_\alpha(x) = x - \alpha(x) \bar{\alpha}^\vee \quad (3.5)$$

is the unique reflection about H_α . Here we put, as usual,

$$\bar{\alpha}^\vee := 2 \frac{\bar{\alpha}}{\|\bar{\alpha}\|^2} \in \bar{\mathfrak{a}}. \quad (3.6)$$

The induced action on $A(\mathfrak{a})$ is then given by

$$s_\alpha(\beta) = \beta - \langle \bar{\beta}, \bar{\alpha}^\vee \rangle \alpha \text{ for all } \beta \in A(\mathfrak{a}). \quad (3.7)$$

3.1. Definition. An *Euclidean reflection group* is a subgroup $W \subseteq M(\mathfrak{a})$ which is generated by reflections and which acts properly on \mathfrak{a} .

Recall, that the action of W is called *proper* if for any compact subset $\Omega \subseteq \mathfrak{a}$ there are only finitely many elements $w \in W$ with $\Omega \cap w\Omega \neq \emptyset$.

A more refined notion is that of an affine root system.

3.2. Definition. A *(reduced) affine root system on \mathfrak{a}* is a subset $\Phi \subset A(\mathfrak{a})$ having the following properties

- a) $\mathbb{R}1 \cap \Phi = \emptyset$.
- b) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.
- c) $\langle \bar{\beta}, \bar{\alpha}^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$
- d) The Weyl group $W_\Phi := \langle s_\alpha \mid \alpha \in \Phi \rangle \subseteq M(\mathfrak{a})$ is an Euclidean reflection group.
- e) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

Remark. Our definition differs from Macdonald's in two respects: First, we assume the root system to be reduced (last axiom). Secondly, we do not assume that $A(\mathfrak{a})$ is spanned by Φ . In fact, Φ may be finite or even empty. So for us all finite root systems are affine, as well.

The classification of affine root systems is well known: First of all there is an essentially unique orthogonal decomposition

$$\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_1 \times \dots \times \mathfrak{a}_n \quad (3.8)$$

such that

$$\Phi = \Phi_1 \cup \dots \cup \Phi_n, \quad (3.9)$$

with $\Phi_\nu \subset \mathfrak{a}_\nu$ irreducible for every $\nu \geq 1$. So each Φ_ν corresponds either to a finite or to a twisted affine Dynkin diagram.

The *chambers of Φ* (or W_Φ) are the connected components of the complement of the union of all reflection hyperplanes in \mathfrak{a} . The closure of a chamber is called an *alcove*. It is known, that W_Φ acts simply transitively on the set of alcoves, that each alcove is a fundamental domain for W_Φ , and that W_Φ is generated by the reflections about the walls (i.e., the faces of codimension one) of any fixed alcove \mathcal{A} . These latter reflections are called *simple* with respect to \mathcal{A} . If Φ is irreducible and infinite then each alcove is a simplex. For finite root systems, alcoves are usually called Weyl chambers and are simplicial cones. In general, an alcove is a direct product of an affine space, a simplicial cone and a number of simplices.

Let $\overline{\Phi}$ be the image of Φ in $\overline{\mathfrak{a}}$. It is a finite, but possibly non-reduced, root system. Its Weyl group $W_{\overline{\Phi}}$ is the image \overline{W}_Φ of W_Φ in $O(\overline{\mathfrak{a}})$.

3.3. Definition. a) Let $\Phi \subset A(\mathfrak{a})$ be an affine root system. A *weight lattice for Φ* is a lattice $\Lambda \subseteq \overline{\mathfrak{a}}$ with $\overline{\Phi} \subset \Lambda$ and $\overline{\Phi}^\vee \subseteq \Lambda^\vee$ where $\Lambda^\vee = \{t \in \overline{\mathfrak{a}} \mid \langle t, \Lambda \rangle \subseteq \mathbb{Z}\}$ is the dual lattice of Λ .

b) An *integral root system* is a pair (Φ, Λ) where $\Phi \subset A(\mathfrak{a})$ is an affine root system and $\Lambda \subseteq \overline{\mathfrak{a}}$ is a weight lattice for Φ .

Let (Φ, Λ) be an integral affine root system on \mathfrak{a} . Then $A := \overline{\mathfrak{a}}/\Lambda^\vee$ is a compact torus. Its character group $\Xi(A) = \text{Hom}(A, U(1))$ can be identified with Λ . More precisely, to $\chi \in \Lambda$ corresponds the character

$$\tilde{\chi} : A \rightarrow U(1) : a + \Lambda^\vee \mapsto e^{2\pi i \langle \chi, a \rangle}. \quad (3.10)$$

For every affine root $\alpha \in \Phi$ we are going to write $\tilde{\alpha} := \widetilde{\overline{\alpha}}$. Dually, every $\eta \in \Lambda^\vee$ defines a cocharacter, namely

$$\tilde{\eta} : U(1) \rightarrow A : e^{2\pi i t} \mapsto t\eta + \Lambda^\vee. \quad (3.11)$$

Again, for $\alpha \in \Phi$ we write $\tilde{\alpha}^\vee := \widetilde{\overline{\alpha}^\vee}$. Then

$$\tilde{\chi}(\tilde{\alpha}^\vee(u)) = u^{\langle \chi, \overline{\alpha}^\vee \rangle} \text{ for all } \chi \in \Lambda, \alpha \in \Phi, u \in U(1). \quad (3.12)$$

In particular,

$$\tilde{\alpha}(\tilde{\alpha}^\vee(u)) = u^2 \text{ for all } \alpha \in \Phi, u \in U(1). \quad (3.13)$$

The Weyl group W_Φ acts on A via its quotient \overline{W}_Φ . More precisely, for $\alpha \in \Phi$ the corresponding reflection acts as

$$s_\alpha(a) = a \cdot \tilde{\alpha}^\vee(\tilde{\alpha}(a))^{-1}, \quad a \in A. \quad (3.14)$$

4. Twisted conjugacy classes

The geometry of twisted conjugacy classes is very well documented in the literature for *simple* groups (to be recalled below). From this, the case of non-simple groups can be easily deduced. For this, *we assume from now on that K is simply connected*. Observe that this means, in particular, that K is semisimple.

We start with a simple observation.

4.1. Lemma. *For $u \in K$ let $\bar{\tau} := \text{Ad}(u) \circ \tau \in \text{Aut } K$, i.e., $\bar{\tau}k = u\tau ku^{-1}$. Let $\varphi : K \rightarrow K : k \mapsto ku^{-1}$. Then*

- a) *The map φ intertwines the τ -twisted action with the $\bar{\tau}$ -twisted action of K on K .*
- b) *Let (M, ω, m) be a quasi-Hamiltonian $K\tau$ -manifold and $\bar{m} := \varphi \circ m : x \mapsto m(x)u^{-1}$. Then (M, ω, \bar{m}) is a quasi-Hamiltonian $K\bar{\tau}$ -manifold.*

Proof. Part a) is an easy calculation. Part b) follows from a) and the easily verified identities $\varphi^*\chi = \chi$, $\varphi^*\Theta_\tau = \Theta_{\bar{\tau}}$. \square

Remark. The Lemma implies that the category of quasi-Hamiltonian $K\tau$ -manifolds depends only on the class of τ in $\text{Out } K = \text{Aut } K / \text{Inn } K$, i.e., on the diagram automorphism which is induced by τ . Thus if we wish, we may assume that τ is induced by a diagram automorphism. On the other hand, we don't want to do that too excessively since sometimes arbitrary automorphisms allow for greater flexibility.

The following facts are well known.

4.2. Theorem. *Let K be a simply connected compact Lie group and τ an automorphism of K . Then there is a τ -stable maximal torus $T \subseteq K$ and an integral affine root system $(\Phi_\tau, \Lambda_\tau)$ on $\mathfrak{a} = \mathfrak{t}^\tau$, the τ -fixed part of $\text{Lie } T$, such that the following holds:*

- a) *Let $\text{pr}^\tau : \mathfrak{t} \rightarrow \mathfrak{a}$ be the orthogonal projection. Then $\bar{\Phi}_\tau = \text{pr}^\tau \Phi(\mathfrak{k}, \mathfrak{t})$ and $\Lambda_\tau = \text{pr}^\tau \Xi(T)$. Moreover, Λ_τ is also the weight lattice (=dual of coroot lattice) of $\bar{\Phi}_\tau$.*
- b) *For any alcove $\mathcal{A} \subseteq \mathfrak{a}$ of Φ_τ the composition*

$$c : \mathcal{A} \hookrightarrow \mathfrak{a} \xrightarrow{\exp} K \rightarrow K\tau/K \quad (4.1)$$

is a homeomorphism.

- c) *For $a \in \mathcal{A}$ let $u := \exp a \in K$. Then the twisted centralizer*

$$K(a) := K_u = \{k \in K \mid ku\tau k^{-1} = u\} = K^{\bar{\tau}} \text{ (where } \bar{\tau} := \text{Ad}(u) \circ \tau) \quad (4.2)$$

is a connected subgroup of K with maximal torus $S := \exp \mathfrak{a} = (T^\tau)^0$. Its root datum is $(\bar{\Phi}_\tau(a), \Lambda_\tau)$ where $\Phi_\tau(a) := \{\alpha \in \Phi \mid \alpha(a) = 0\}$.

For a proof see e.g. [Seg68], [Moh03], or [MW04]. For the sake of computing examples, let me indicate the construction of Φ_τ .

The automorphism τ permutes the simple factors of K . Thus there exists a τ -stable decomposition $K = K_1 \times \dots \times K_s$ such that $\langle \tau \rangle$ acts transitively on the simple factors of each K_i . Let $\tau_i := \text{res}_{K_i}(\tau)$. Suppose that $\Phi_{\tau_i} \subseteq \mathfrak{a}_i = \mathfrak{t}_i^{\tau_i}$ is already constructed. Then

$$\Phi_\tau := \Phi_{\tau_1} \cup \dots \cup \Phi_{\tau_s} \subseteq \mathfrak{a} := \mathfrak{a}_1 \oplus \dots \oplus \mathfrak{a}_s. \quad (4.3)$$

We are now reduced to the case that $\langle \tau \rangle$ permutes the factors of K transitively. This means that $K \cong K_0^m$ with K_0 simple and there is $\tau_0 \in \text{Aut}(K_0)$ such that τ acts on K as

$$\tau(k_1, k_2, \dots, k_m) = (k_2, \dots, k_m, \tau_0 k_1). \quad (4.4)$$

The twisted action on K_0^m is

$$(k_1 g_1 k_2^{-1}, \dots, k_{m-1} g_{m-1} k_m^{-1}, k_m g_m \tau_0 k_1^{-1}). \quad (4.5)$$

Therefore, $1 \times K_0^{m-1} \subseteq K$ acts freely on K with quotient map

$$K_0^m \rightarrow K_0\tau_0 : (g_1, \dots, g_m) \mapsto g_1 \dots g_m \quad (4.6)$$

which is equivariant with respect to the first copy of K_0 . Let $\Phi_{\tau_0} \subset \mathfrak{a}_0$ be the affine root system for τ_0 . Then $\mathfrak{a} = \mathfrak{a}_0$ is embedded diagonally into \mathfrak{a}_0^m and Φ_τ consists of all affine linear functions of the form $\alpha(x) = \frac{1}{m}\alpha_0(mx)$ with $\alpha_0 \in \Phi_{\tau_0}$. Observe though that the scalar product on \mathfrak{a} differs from that on \mathfrak{a}_0 by a factor of m .

So we may assume that K is simple. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra, $\Phi_K \subseteq \mathfrak{t}^*$ the corresponding root system, $\mathfrak{t}_+ \subseteq \mathfrak{t}$ a Weyl chamber, and $\Phi_K^+ \subseteq \Phi_K$ the corresponding set of positive roots. Lemma 4.1 allows us to assume that τ is induced by a graph automorphism. Let $\mathfrak{a} = \mathfrak{t}^\tau = \{\xi \in \mathfrak{t} \mid \tau\xi = \xi\}$ be the space of τ -fixed points in \mathfrak{t} . Then the set $\overline{\Phi}_\tau := \{\text{pr}^\tau \alpha \mid \alpha \in \Phi(\mathfrak{k}, \mathfrak{t})\}$ of restricted roots is a (possibly not reduced) root system on \mathfrak{t}^τ . Let $\overline{S}_\tau \subseteq \overline{\Phi}_\tau$ be the set of simple roots with respect to the Weyl chamber $\mathfrak{t}^\tau \cap \mathfrak{t}^+$.

Let $r \in \{1, 2, 3\}$ be the order of τ . Then we define $\theta \in \overline{\Phi}_\tau$ to be the longest dominant root if $r = 1$ or $r = 2$ and $K \cong SU(2n+1)$ (case $A_{2n}^{(2)}$). Otherwise, θ denotes the dominant short root of $\overline{\Phi}_\tau$. Define the affine linear function $\alpha_0(x) = -\langle \theta, x \rangle + \frac{2\pi}{r}$. Then Φ_τ is the affine root system whose set of simple roots is

$$S_\tau := \overline{S}_\tau \cup \{\alpha_0\}. \quad (4.7)$$

A first application of the description of twisted conjugacy classes goes as follows. Let $m : M \rightarrow K\tau$ be a moment map and $\mathcal{A} \subseteq \mathfrak{a}$ an alcove. Since $c : \mathcal{A} \rightarrow K\tau/K$ (see (4.1)) is bijective one can define the *invariant moment map* as

$$m_+ := c^{-1} \circ m : M \rightarrow \mathcal{A}. \quad (4.8)$$

This yields a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{m} & K \\ m_+ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow[\text{bij}]{c} & K\tau/K \end{array} \quad (4.9)$$

Observe that c is in general just continuous but not smooth, so the same holds true for m_+ .

4.3. Definition. Let $m : M \rightarrow K\tau$ be quasi-Hamiltonian. Then $\mathcal{P}_M = m_+(M) \subseteq \mathcal{A}$ is called the *momentum image* of M .

Observe that \mathcal{P}_M determines the actual image of m since $m(M) = K \cdot \exp \mathcal{P}_M$. Fundamental is the following

4.4. Theorem ([AMM98, Thm. 7.2], [Mei15, Thm. 4.4]). *Let K be simply connected and let (M, m) be a connected compact quasi-Hamiltonian $K\tau$ -manifold with moment map $m : M \rightarrow K\tau$. Then its momentum image \mathcal{P}_M is a convex polytope lying inside \mathcal{A} . Moreover, all fibers of m (and therefore m_+) are connected.*

Since we want to glue compact multiplicity free manifolds from local pieces we have to weaken the compactness property.

4.5. Definition. A multiplicity free quasi-Hamiltonian K -manifold M is called *convex* if its momentum image \mathcal{P}_M is convex and locally closed in \mathcal{A} .

For example, if M is compact and multiplicity free and $U \subseteq \mathcal{A}$ is any convex open subset then $M_U := m_+^{-1}(U)$ is multiplicity free and convex in the above sense.

5. The local structure of quasi-Hamiltonian manifolds

The local structure of the space of twisted conjugacy classes is also well known from, e.g., [Moh03, MW04]. Let $a \in \mathcal{A}$ (notation as in Theorem 4.2) and $u := \exp(a) \in K$. Let

$$L := \{l \in K \mid lu \tau^{-1} = u\} = \{l \in K \mid \tau l = u^{-1}lu\} \quad (5.1)$$

be its (twisted) stabilizer in K . Observe that $\tau u = u$ implies $u \in L$ and $\tau L = L$. Then an easy calculation shows that

$$\varphi : L \rightarrow K : l \mapsto lu \quad (5.2)$$

is an L -equivariant map with $\varphi(e) = u$ where L acts on itself and K by untwisted and τ -twisted conjugation, respectively. Let $\bar{\tau} := \text{Ad}(u) \circ \tau$ and let $\mathcal{O} \subseteq K$ be the twisted conjugacy class of u . Then another easy calculation gives for its tangent space

$$T_u \mathcal{O} = \{(\xi - \bar{\tau}\xi)u \mid \xi \in \mathfrak{k}\} = {}^{1-\bar{\tau}}\mathfrak{k} u \quad (5.3)$$

On the other hand

$$\text{Lie } L = \ker(1 - \bar{\tau}) \subseteq \mathfrak{k}. \quad (5.4)$$

This shows that $\varphi(L)$ is a slice for \mathcal{O} in u . Now let $U \subseteq \mathcal{A}$ be an open neighborhood of a which is small enough such that $U_0 := U - a$ is open in the cone $\mathcal{C} := \mathbb{R}_{\geq 0}(\mathcal{A} - a)$. Observe that \mathcal{C} is a Weyl chamber of L by Theorem 4.2. Then $L_U := \text{Ad } L U_0$ is an open neighborhood of $1 \in L$. This determines the local structure of $K\tau$ near \mathcal{O} :

5.1. Lemma. *The map*

$$\Phi : K \times^L L_U \rightarrow K\tau : [k, l] \mapsto k\varphi(l) \tau k^{-1} = klu \tau k^{-1} \quad (5.5)$$

is a K -equivariant diffeomorphism onto an open neighborhood of \mathcal{O} in K .

Now let $m : M \rightarrow K\tau$ be a quasi-Hamiltonian manifold. Then the pull-back with φ yields the L -manifold $M_U = M \times_K L_U$ such that the following diagram commutes

$$\begin{array}{ccc} M_U & \hookrightarrow & M \\ \downarrow m_L & & \downarrow m \\ L & \xrightarrow{\varphi} & K \end{array} \quad (5.6)$$

(where m_L has in fact values in L_U). Then in [AMM98] it was shown that M_U carries canonically the structure of a quasi-Hamiltonian L -manifold with moment map m_L and 2-form $\omega_L = \omega|_{M_U}$. More generally, the following holds:

5.2. Proposition. *Every $a \in \mathcal{A}$ has an open neighborhood $U \subseteq \mathcal{A}$ such that the functor $M \mapsto M_U$ is an equivalence between the category of quasi-Hamiltonian $K\tau$ -manifolds (M, m) with $m_+(M) \subseteq U$ and (untwisted) quasi-Hamiltonian L -manifolds (M', m') with $m'_+(M') \subseteq U - a$.*

Proof. Lemma 4.1 with $u = \exp a$ allows to replace τ by $\bar{\tau} = \text{Ad}(u) \circ \tau$. Thereby, we may assume that $a = 0$. We proceed to construct a functor which is quasi-inverse to $M \mapsto M_U$. For this, recall from [AMM98] the double $D(K)$ of K . It is a quasi-Hamiltonian $K \times K$ -manifold which equals $K \times K$ as a manifold. The action of $K \times K$ on $D(K)$ is given by

$$(k_1, k_2) * (b_1, b_2) = (k_1 b_1 k_2^{-1}, k_2 b_2 k_1^{-1}). \quad (5.7)$$

The moment map is

$$m_{D(K)} : D(K) \rightarrow K \times K : (b_1, b_2) \mapsto (b_1 b_2, b_1^{-1} b_2^{-1}). \quad (5.8)$$

The *twisted double* is the open subset $D_\tau(K) = K \times K\tau$ of $D(\mathbb{Z}\tau \ltimes K)$. By identifying $D_\tau(K)$ with K^2 we get the twisted action

$$(k_1, k_2) * (b_1, b_2) = (k_1 b_1 k_2^{-1}, k_2 b_2 \tau k_1^{-1}). \quad (5.9)$$

The moment map takes values in $K\tau \times \tau^{-1}K$. After identifying also that space with K^2 we get the $\tau \times \tau^{-1}$ -twisted moment map

$$m_{D_\tau(K)} : D_\tau(K) \rightarrow K \times K : (b_1, b_2) \mapsto (b_1 b_2, \tau b_1^{-1} b_2^{-1}). \quad (5.10)$$

Next put $L_0 := \text{Ad } L(\exp(U - a))$. It follows from Theorem 4.2 that L_0 is a conjugation invariant open neighborhood of $1 \in L$. Now put

$$Z := D_\tau(K)_{K \times L_0^{-1}} = \{(b_1, b_2) \in K \times K \mid b_2 \tau b_1 \in L_0\}. \quad (5.11)$$

This is a quasi-Hamiltonian $K\tau \times L$ -manifold which we can identify with $K \times L_0$ via the map

$$K \times L_0 \xrightarrow{\sim} Z : (b, c) \mapsto (b, c \tau b^{-1}). \quad (5.12)$$

Because of $\tau l = l$ for all $l \in L$, the induced $K \times L$ -action on $K \times L_0$ is

$$(k, l) * (b, c) = (k b l^{-1}, l c l^{-1}). \quad (5.13)$$

while the moment map becomes

$$m_Z : K \times L_0 \rightarrow K \times L : (b, c) \mapsto (b c \tau b^{-1}, c^{-1}). \quad (5.14)$$

In [AMM98] also the *fusion* of two quasi-Hamiltonian manifolds was introduced. More precisely, let M_1 be a quasi-Hamiltonian $K \times L$ -manifold with moment map (m_1, m_2) and let M_2 be a quasi-Hamiltonian L -manifold with moment map m_3 . Then it was shown that $M_1 \times M_2$ is a quasi-Hamiltonian $K \times L$ -manifold with action $(k, l)(x_1, x_2) = (k x_1 l^{-1}, l x_2)$ and moment map $(x_1, x_2) \mapsto (m_1(x_1), m_2(x_1) m_3(x_2))$. We are going to denote this new manifold by $M_1 \otimes_L M_2$. This construction also works if the action of K is twisted.

Finally, recall from [AMM98] also the process of *symplectic reduction* of a quasi-Hamiltonian $K \times L$ manifold M with moment map (m_1, m_2) . Assume that $1 \in L$ is a regular value for m_2 . Then it is shown that

$$M // L := m_2^{-1}(1) / L \quad (5.15)$$

is a quasi-Hamiltonian K -manifold with moment map $Lx \mapsto m_1(x)$. Also this works for a twisted K -action.

Now the desired quasi-inverse functor is

$$\text{ind}_L^K M' := (Z \otimes_L M') // L \quad (5.16)$$

which is quasi-Hamiltonian by construction. Moreover, by definition

$$\text{ind}_L^K M' = \{(b, c, x) \in K \times L_0 \times M' \mid c^{-1} m'(x) = 1\} / L \cong (K \times M') / L = K \times^L M'. \quad (5.17)$$

An easy calculation shows that the moment map induces on $K \times^L M'$ the map

$$K \times^K M' \rightarrow K : [b, x] \mapsto b m'(x) \tau b^{-1}. \quad (5.18)$$

Thus, it suffices to show that the natural maps

$$\varphi : M' \rightarrow K \times^L M' : x \mapsto [1, x] \quad (5.19)$$

and

$$\psi : K \times^L M_U \rightarrow M : [k, x] \mapsto kx \quad (5.20)$$

are isomorphisms of quasi-Hamiltonian manifolds. First, it follows immediately from Lemma 5.1 that both φ and ψ are K -equivariant diffeomorphisms which are compatible with the moment maps. It remains to show that the 2-forms match up.

For φ observe that $x \in M'$ is mapped to the class of $(1, m(x), x) \in K \times L_0 \times M'$. Now recall the explicit formula of [AMM98, Thm. 6.1] for the 2-form on the fusion product of two quasi-Hamiltonian manifolds (M_1, ω_1, m_1) and (M_2, ω_2, m_2) :

$$\pi_1^* \omega_1 + \pi_2^* \omega_2 + \frac{1}{2} \langle m_1^* \theta, m_2^* \bar{\theta} \rangle. \quad (5.21)$$

which we apply to $M_1 = Z$ and $M_2 = M'$. Let $\iota : L \rightarrow L : h \mapsto h^{-1}$ be the inversion. Because of $\iota^* \theta = -\bar{\theta}$ the pull-back of the third term in (5.21) to M' vanishes. To see that also the first summand vanishes on M' we look at the explicit form of ω_1 on $D(K)$ (see [AMM98, Prop. 3.2]):

$$\omega_1 = \frac{1}{2} \langle p_1^* \theta, p_2^* \bar{\theta} \rangle + \frac{1}{2} \langle p_1^* \bar{\theta}, p_2^* \theta \rangle \quad (5.22)$$

where p_1, p_2 are the two projections of $D(K)$ to K . The map from M' to $Z \subseteq D(K)$ is $x \mapsto (1, m'(x))$. Hence p_1 is constant on M' implying that the pull-backs of $p_1^* \theta$ and $p_1^* \bar{\theta}$, hence of ω_1 to M' vanish. This finishes the proof that φ is an isomorphism of quasi-Hamiltonian manifolds.

To show this for ψ let ω be the given 2-form on M . We claim that ω is uniquely determined by the moment map property of m and its restriction to M_U . By K -invariance, ω is determined by its values in any $x \in M_U$. The moment map property Definition 2.3c allows to compute $\omega(\xi, \eta)$ where $\xi \in \mathfrak{k}_x$ and $\eta \in T_x M$. Moreover, also $\omega(\xi, \eta)$ is known for $\xi, \eta \in T_x M_U$. Because of $\mathfrak{k}_x + T_x M_U = T_x M$ we proved our claim. Now let ω' be the 2-form on $K \times^L M_U$. Then the claim shows $(\psi^{-1})^* \omega' = \omega$ which is what had to be proved. \square

Now we pull everything back to the Lie algebra \mathfrak{l} of L using the exponential map. For this let $\mathfrak{l}_0 := \text{Ad } L(U - a) \subseteq \mathfrak{l}$. It is an L -invariant open neighborhood of $0 \in \mathfrak{l}$ such that $\exp : \mathfrak{l}_0 \rightarrow L_0$ is a diffeomorphism. We denote the inverse of this diffeomorphism by \log_U .

Now assume that (M, ω, m_0) is a Hamiltonian manifold in the ordinary sense. This means in particular that m_0 is an L -equivariant map from M to \mathfrak{l}^* which we continue to identify with \mathfrak{l} . The cone spanned by $U - a$ in \mathfrak{t} is a Weyl chamber \mathfrak{t}^+ for \mathfrak{l} . Hence we get a homeomorphism $\mathfrak{t}^+ \rightarrow \mathfrak{l} // L$. Inverting it, one can also define the invariant moment map

$$(m_0)_+ : M \rightarrow \mathfrak{l} \rightarrow \mathfrak{l} // L \xrightarrow{\sim} \mathfrak{t}^+. \quad (5.23)$$

To get a Hamiltonian manifold from a quasi-Hamiltonian one, one defines the 2-form $\tilde{\omega}$ on \mathfrak{l} by

$$\tilde{\omega}_\lambda(\xi_1, \xi_2) = \langle g(\text{Ad } \lambda) \xi_1, \xi_2 \rangle \quad (5.24)$$

where $\lambda, \xi_1, \xi_2 \in \mathfrak{l}$ and

$$g(x) := \frac{\sinh x - x}{x^2} = \frac{x}{3!} + \frac{x^3}{5!} + \frac{x^5}{7!} + \dots \quad (5.25)$$

An easy calculation shows that this two-form equals the two form ϖ in Lemma 3.3 of [AMM98]. Now for a quasi-Hamiltonian L -manifold (M, ω, m) with $m(M) \subseteq U$ we put $m_0 := \log_U \circ m : M \rightarrow \mathfrak{l} \cong \mathfrak{l}^*$ and $\omega_0 := \omega - m_0^* \tilde{\omega}$.

5.3. Lemma. *The functor $(M, \omega, m) \mapsto \log M := (M, \omega_0, m_0)$ is an equivalence between the category of quasi-Hamiltonian L -manifolds (M, ω, m) with $m_+(M) \subseteq U$ and Hamiltonian L -manifolds (M_0, m_0) with $(m_0)_+(M_0) \subseteq U$.*

Proof. This is [AMM98, Prop. 3.4 and Rem. 3.3]. The quasi-inverse functor is

$$(M_0, \omega_0, m_0) \mapsto \exp M_0 := (M_0, \omega_0 + m_0^* \tilde{\omega}, \exp \circ m_0) \quad \square$$

Putting both constructions together we get:

5.4. Theorem. *Any $a \in \mathcal{A}$ has an open neighborhood $U \subseteq \mathcal{A}$ such that there is an equivalence between the category of quasi-Hamiltonian $K\tau$ -manifolds (M, m) with $m_+(M) \subseteq U$ and Hamiltonian L -manifolds (M_0, m_0) with $(m_0)_+(M_0) \subseteq U - a$.*

Remarks. *i)* The theorem hold only if the manifolds are allowed to be non-connected. The reason for this is that even if M is connected, the local model M_0 might be not.

ii) It can be shown that the only requirement for the open subset U of \mathcal{A} is the validity of the Slice Lemma 5.1. As already remarked in [AMM98, Rem. 7.1] this implies that for any L there is a canonical open set U . More precisely, let $\mathcal{A}^\sigma \subseteq \mathcal{A}$ be a face of \mathcal{A} . Then $\mathcal{A}_\sigma \subseteq \mathcal{A}$ is obtained by removing all faces which do not contain \mathcal{A}^σ :

$$\mathcal{A}_\sigma := \mathcal{A} \setminus \bigcup_{\mathcal{A}^\sigma \not\subseteq \mathcal{A}^\eta} \mathcal{A}^\eta \quad (5.26)$$

This is an open subset of \mathcal{A} . Now choose $a \in \mathcal{A}^\sigma \cap \mathcal{A}_\sigma$. Then the stabilizer $K_\sigma = K_a$ is independent of the choice of a . Then Theorem 5.4 holds for $L = K_\sigma$ and $U = \mathcal{A}_\sigma$.

Later, we need the property that the equivalence of categories above is compatible with Hamiltonian dynamics. For this, let f be a K -invariant smooth function on M . Then it was shown in [AMM98, Prop. 4.6] that M carries a unique vector field H_f with

$$\iota(H_f)\omega = df \text{ and } \iota(H_f)m^*\theta = 0. \quad (5.27)$$

Observe, that the second condition is only necessary when ω is degenerate. Now let $f_0 := f|_{M_U}$ which can be considered as an L -invariant function on $M_0 = M_U$. It induces a Hamiltonian vector field in the classical sense:

$$\iota(H_{f_0})\omega_0 = df_0. \quad (5.28)$$

5.5. Lemma. *Let f be a K -invariant smooth function on M and f_0 its restriction to M_U . Then H_{f_0} coincides with H_f on M_U . In particular, the Hamiltonian flow generated by f on M preserves M_U and coincides there with the one generated by f_0 on M_0 .*

Proof. The vector field H_f is parallel to the fibers of m . Since $m = \exp \circ m_0$, it is on M_0 also parallel to the fibers of m_0 . Hence it lies in the kernel of $m_0^* \tilde{\omega}$. This implies

$$\iota(H_f)\omega_0 = \iota(H_f)\omega|_{M_U} = df|_{M_U} = df_0 \quad (5.29)$$

which is the characterizing equation of H_{f_0} . \square

6. Classification of multiplicity free quasi-Hamiltonian manifolds

In this section we describe our main classification result in detail. The proof will be given in the subsequent sections.

In general, the moment map can have quite pathological properties. See e.g. [Kno02]. If the manifold is compact these pathologies won't occur because of the following

6.1. Theorem. *Assume K to be simply connected and let M be a compact quasi-Hamiltonian manifold. Then $m_+(M)$ is a convex subset of \mathcal{A} and $m_+/K : M/K \rightarrow m_+(M)$ is a homeomorphism.*

Proof. By [AMM98, Thm. 7.2], the image $m_+(M)$ is convex and the fibers of m and therefor of m_+/K are connected. On the other hand, these fibers are discrete since M is multiplicity free. Hence m_+/K is bijective and hence, by compactness, a homeomorphism. \square

The conclusion of the theorem is inherited by M_U whenever the open set U itself is convex. Since we want to glue M from local pieces, we define:

6.2. Definition. A quasi-Hamiltonian manifold (M, m) is called *convex and multiplicity free* if

- a) The momentum image $\mathcal{P}_M := m_+(M)$ is a convex subset of \mathcal{A} .
- b) The map $m_+/K : M/K \rightarrow \mathcal{P}_M$ is a homeomorphism.

Now let $a \in \mathcal{P}_M$ and $U \subseteq \mathcal{A}$ such that the conclusion of the local structure theorem holds. Without loss of generality we may assume that U is convex. Then M_0 is a convex multiplicity free Hamiltonian L -manifold. For its momentum image holds

$$\mathcal{P}_{M_0} = \mathcal{P}_M \cap U. \quad (6.1)$$

Observe that *b)* implies that the fibers of m_+ are K -orbits, hence connected. Therefore M_0 is connected, as well. Recall the space \mathfrak{a} from Theorem 4.2. Then we define the *tangent cone* of a subset $\mathcal{P} \subseteq \mathcal{A}$ in $a \in \mathcal{P}$ as

$$C_a \mathcal{P} := \mathbb{R}_{\geq 0}(\mathcal{P} - a) \subseteq \mathfrak{a}. \quad (6.2)$$

The discussion above and [Kno02, Thm. 2.7] imply the following structural property of \mathcal{P}_M .

6.3. Lemma. *Let M be a convex, multiplicity free quasi-Hamiltonian $K\tau$ -manifold. Then \mathcal{P}_M is a locally polyhedral set, i.e., for every $a \in \mathcal{P}_M$ the tangent cone $C_a := C_a \mathcal{P}_M$ is a finitely generated convex cone and there is an open neighborhood U of a in \mathcal{A} with*

$$\mathcal{P}_M \cap U = C_a \cap U. \quad (6.3)$$

From this we get

6.4. Corollary. *Let $\mathfrak{a}_M \subseteq \mathfrak{a}$ be the affine subspace spanned by \mathcal{P}_M . Then the interior \mathcal{P}_M^0 of \mathcal{P}_M in \mathfrak{a}_M is non-empty and dense in \mathcal{P}_M .*

The dimension of \mathfrak{a}_M is an important invariant of M , called the *rank* $\text{rk } M$.

Next, we study the generic isotropy group of M . Lemma 2.3 of [Kno11] combined with Theorem 5.4 (reduction to the local case) implies

6.5. Lemma. *Let $a \in \mathcal{P}_M^0$ and let $L_0 \subseteq L$ be the kernel of the L -action on M_0 . Then $\bar{A}_M := L/L_0$ is a torus and L_0 is a principal isotropy group for K on M .*

The group L_0 can be encoded by a lattice as follows. Since $\mathfrak{l} = \mathfrak{a} + \mathfrak{l}_0$, the orthogonal complement $\bar{\mathfrak{a}}_M$ of $\mathfrak{a} \cap \mathfrak{l}_0$ in \mathfrak{a} can be identified with $\text{Lie } A_M$. It follows from the properties of a moment map that $\bar{\mathfrak{a}}_M$ is the group of translations of the affine space \mathfrak{a}_M . Indeed let $\tilde{\mathfrak{a}}_M := \mathfrak{a}_M - \mathfrak{a}_M$. Then

$$\xi \in \tilde{\mathfrak{a}}_M^\perp \Leftrightarrow \langle \xi, \cdot \rangle \text{ is constant on } \mathfrak{a}_M \Leftrightarrow \xi_* = 0 \Leftrightarrow \xi \in \mathfrak{a} \cap \mathfrak{l}_0 \Leftrightarrow \xi \in \bar{\mathfrak{a}}_M^\perp. \quad (6.4)$$

This shows that the Lie algebra \mathfrak{l}_0 is already determined by the momentum image \mathcal{P}_M . For the group itself, one needs additionally the lattice

$$\Lambda_M^\vee := \ker \exp : \bar{\mathfrak{a}}_M \rightarrow A_M. \quad (6.5)$$

In the following, we prefer to work with its dual lattice

$$\Lambda_M := \{x \in \bar{\mathfrak{a}}_M \mid \langle x, \Lambda_M^\vee \rangle \subseteq \mathbb{Z}\} \subseteq \bar{\mathfrak{a}}_M \quad (6.6)$$

which can be also interpreted as the character group $\Xi(A_M)$ of A_M . We call Λ_M the *character group* of M .

We are going to classify multiplicity free manifolds in terms of the pair $(\mathcal{P}_M, \Lambda_M)$. To describe which pairs are possible we need some notions from algebraic geometry.

Let $G = K_{\mathbb{C}}$ be the complexification of K . This is a connected complex reductive group. Let $B \subseteq G$ be a Borel subgroup and let $\mathcal{X}(B) := \text{Hom}(B, \mathbb{C}^*) \cong \mathbb{Z}^{\text{rk } G}$ be its character group. It is possible to identify $\mathcal{X}(B) \otimes \mathbb{R}$ with a Cartan subalgebra \mathfrak{t} of \mathfrak{k} (actually the dual of one). The characters which lie in the Weyl chamber \mathfrak{t}^+ are called dominant. Recall, that there is a 1 : 1-correspondence $\chi \mapsto L(\chi)$ between dominant characters and irreducible representations of G .

A G -variety X is called *spherical* if B has an open dense orbit in X . In the following we are only interested in affine varieties. In this case, there is a purely representation theoretic criterion for sphericity due to Vinberg-Kimel'fel'd, [VK78]: Let $\mathbb{C}[X]$ be the ring of regular functions on X . Then $\mathbb{C}[X]$ is in particular a representation of G and decomposes as a direct sum of irreducible representation. The criterion states that X is spherical if and only if $\mathbb{C}[X]$ is a multiplicity free module for G , i.e., no irreducible representation appears in $\mathbb{C}[X]$ more than once. Under these conditions there is a subset $\Lambda_X^+ \subseteq \mathcal{X}(B) \cap \mathfrak{t}^+$ such that

$$\mathbb{C}[X] = \bigoplus_{\chi \in \Lambda_X^+} L(\chi) \quad (6.7)$$

as a G -representation. The set Λ_X^+ is actually additively closed and is called the *weight monoid* of X .

Now we return to the compact group K . Let $\mathcal{P} \subseteq \mathcal{A}$ be a locally closed convex subset. Let $\mathfrak{a}_{\mathcal{P}} \subseteq \mathfrak{t}$ be the affine space spanned by \mathcal{P} and let

$$\bar{\mathfrak{a}}_{\mathcal{P}} := \mathfrak{a}_{\mathcal{P}} - \mathfrak{a}_{\mathcal{P}} \quad (6.8)$$

be its group of translations. A point $a \in \mathcal{P}$ gives also rise to an element $u = \exp(a) \in K\tau$ and hence to an isotropy group $K(a)$ with respect to the twisted action. Observe that the tangent cone of \mathcal{A} in a is a Weyl chamber for $K(a)$.

6.6. Definition. Let K be a simply connected compact Lie group with automorphism τ and fundamental alcove \mathcal{A} . Let $\mathcal{P} \subseteq \mathcal{A}$ be a locally closed convex subset and $\Lambda \subseteq \bar{\mathfrak{a}}_{\mathcal{P}}$ a lattice. Then (\mathcal{P}, Λ) is called *spherical in* $a \in \mathcal{P}$ if

a) \mathcal{P} is polyhedral in a , i.e.,

$$\mathcal{P} \cap U = (a + C_a \mathcal{P}) \cap U \quad (6.9)$$

for a neighborhood U of a in \mathcal{A} , and

b) there is a smooth affine spherical $K(a)_{\mathbb{C}}$ -variety X such that

$$C_a \mathcal{P} \cap \Lambda = \Lambda_X^+. \quad (6.10)$$

The pair (\mathcal{P}, Λ) is *spherical* if it is spherical in all $a \in \mathcal{P}$.

Here is our main result:

6.7. Theorem. *Let K be a simply connected compact Lie group with twist τ . Then the map $M \mapsto (\mathcal{P}_M, \Lambda_M)$ furnishes a bijection between*

- *isomorphism classes of convex multiplicity free quasi-Hamiltonian $K\tau$ -manifolds and*
- *spherical pairs (\mathcal{P}, Λ_M) .*

Under this correspondence, M is compact if and only if \mathcal{P}_M is closed in \mathcal{A} .

In the remainder of this section we reduce the proof of the main Theorem 6.7 to a statement about automorphisms. We start with:

6.8. Lemma. *Let M be a convex multiplicity free quasi-Hamiltonian $K\tau$ -manifold. Then the pair $(\mathcal{P}_M, \Lambda_M)$ is spherical.*

Proof. Using the cross section theorem 5.4, the problem is reduced to the Hamiltonian case. Then it is part of [Kno11, Thm. 11.2]. \square

Next, we state local existence and uniqueness:

6.9. Lemma. *Let (\mathcal{P}, Λ) be a spherical pair.*

- a) *For every point $a \in \mathcal{P}$ there is a convex multiplicity free quasi-Hamiltonian $K\tau$ -manifold M with $\Lambda_M = \Lambda$ and such that \mathcal{P}_M is an open neighborhood of a in \mathcal{P} .*
- b) *Let $(M, m), (M', m')$ be two convex multiplicity free quasi-Hamiltonian $K\tau$ -manifolds with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}_{M'}, \Lambda_{M'}) = (\mathcal{P}, \Lambda)$. Then there is an open cover $\mathcal{P} = \bigcup_{\nu} \mathcal{P}_{\nu}$ and for all ν isomorphisms of quasi-Hamiltonian $K\tau$ -manifolds*

$$m_+^{-1}(\mathcal{P}_{\nu}) \cong (m')_+^{-1}(\mathcal{P}_{\nu}). \quad (6.11)$$

Proof. Again, by localizing, we may assume that M is Hamiltonian. Then a) holds by the definition of a spherical pair and b) is basically a result of Losev [Los09] (see also [Kno11, Thm. 2.4]). \square

Next, we consider automorphisms.

6.10. Lemma. *Let M be a convex, multiplicity free Hamiltonian or quasi-Hamiltonian manifold. Then its automorphism group is abelian.*

Proof. By localization, one can assume that M is Hamiltonian. Then use [Kno11, Thm. 9.2]. \square

This can be used as follows: Let (\mathcal{P}, Λ) be a spherical pair and let $\mathcal{P}_0 \subseteq \mathcal{P}$ be open such that there is a convex multiplicity free manifold M with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}_0, \Lambda)$. Then

$$\mathfrak{L}_{\mathcal{P}, \Lambda}(\mathcal{P}_0) := \text{Aut } M \quad (6.12)$$

depends only on \mathcal{P}_0 and not on the choice of M . This follows from the fact that any isomorphism $M \cong M'$ induces an isomorphism of the automorphism groups which is unique up to conjugation, so canonical because of abelianness.

Since $\mathfrak{L}_{\mathcal{P}, \Lambda}$ is a sheaf of abelian groups it has cohomology groups. Now, the technical heart of this paper is:

6.11. Theorem. *Let (\mathcal{P}, Λ) be a spherical pair. Then $H^i(\mathcal{P}, \mathfrak{L}_{\mathcal{P}, \Lambda}) = 0$ for all $i \geq 1$.*

The proof will be given in the next sections. See Theorem 9.5 for a description of $\mathfrak{L}_{\mathcal{P}, \Lambda}$ and Theorem 10.1 for the cohomology vanishing.

Now we can prove the main Theorem 6.7: the vanishing of H^2 implies that the local models from Lemma 6.9a) glue to a global model M . The vanishing of H^1 implies that all local isomorphisms from Lemma 6.9b) glue to a global isomorphism.

Remark. In modern parlance, the argument goes as follows: Let (\mathcal{P}, Λ) be a spherical pair and let $\mathbf{MF}_{\mathcal{P}, \Lambda}$ be the category of (locally) convex multiplicity free quasi-Hamiltonian manifolds M with $\Lambda_M = \Lambda$ and $\mathcal{P}_M \subseteq \mathcal{P}$ open. Then Lemma 6.9 means that $\mathbf{MF}_{\mathcal{P}, \Lambda}$ is a *gerbe* over \mathcal{P} . Since all automorphism groups are abelian, its *band* $\mathfrak{L}_{\mathcal{P}, \Lambda}$ is a sheaf of abelian groups. Now $H^2(\mathcal{P}, \mathfrak{L}_{\mathcal{P}, \Lambda}) = 0$ means that $\mathbf{MF}_{\mathcal{P}, \Lambda}$ is equivalent to the category $\text{Tors } \mathfrak{L}_{\mathcal{P}, \Lambda}$ of $\mathfrak{L}_{\mathcal{P}, \Lambda}$ -torsors. Because of the vanishing of H^1 , all torsors over \mathcal{P} are trivial. So $\mathbf{MF}_{\mathcal{P}, \Lambda}$ contains up to isomorphism exactly one object M with $\mathcal{P}_M = \mathcal{P}$.

7. The automorphism group of a multiplicity free manifold

Return to the notation of section 3, i.e., $\bar{\mathfrak{a}}$ is an Euclidean vector space, \mathfrak{a} is an affine space for $\bar{\mathfrak{a}}$, and Φ is a root system on \mathfrak{a} with fundamental alcove $\mathcal{A} \subseteq \mathfrak{a}$. Let moreover $\Lambda \subseteq \bar{\mathfrak{a}}$ be a weight lattice for Φ . This defines the torus $A := \bar{\mathfrak{a}}/\Lambda^\vee$.

Let $\mathcal{P} \subseteq \mathcal{A}$ be a locally polyhedral convex subset with non-empty interior \mathcal{P}^0 . This means, in particular, that \mathcal{P}° is dense in \mathcal{P} . We proceed to define a number of properties that a map $\varphi : \mathcal{P} \rightarrow A$ may have.

- a) A map $\varphi : \mathcal{P} \rightarrow A$ is called *smooth* if for every $x \in \mathcal{P}$ there is a smooth map $\tilde{\varphi} : U \rightarrow A$ where U is an open neighborhood of $x \in \mathfrak{a}$ and $\tilde{\varphi}|_{\mathcal{P} \cap U} = \varphi|_{\mathcal{P} \cap U}$. Let $\hat{\mathcal{C}}_{\mathfrak{a}, x}$ and $\hat{\mathcal{C}}_{A, a}$ be the completions of the local ring of smooth functions (i.e., formal power series) in $x \in \mathfrak{a}$ and $a \in A$, respectively. Then if φ is smooth with $\varphi(x) = a$ it induces an algebra homomorphism $\hat{\varphi}_x : \hat{\mathcal{C}}_{A, \varphi(x)} \rightarrow \hat{\mathcal{C}}_{\mathfrak{a}, x}$. In fact, by continuity, $\hat{\varphi}_x = \hat{\varphi}_x$ is independent of the choice of $\tilde{\varphi}$ since \mathcal{P}^0 is dense in \mathcal{P} .

- b) The scalar product on $\bar{\mathfrak{a}}$ induces a canonical symplectic structure on the product spaces of $\mathfrak{a} \times A$ by

$$\omega(\xi_1 + \eta_1, \xi_2 + \eta_2) = \langle \xi_1, \eta_2 \rangle - \langle \xi_2, \eta_1 \rangle. \quad (7.1)$$

After the identifications $\mathfrak{a} \cong \bar{\mathfrak{a}} \cong \bar{\mathfrak{a}}^*$ this is just the canonical symplectic form on the cotangent bundle T_A^* . Now, a smooth map $\varphi : \mathcal{P} \rightarrow A$ is called *closed* if the graph of $\varphi|_{\mathcal{P}^0}$ is a Lagrangian submanifold of $\mathfrak{a} \times A$.

- c) Recall that the Weyl group W of Φ fixes the lattice Λ and therefore acts on A by way of group automorphisms. Let W_0 be a subgroup of W and for any $x \in \mathcal{P}$ let $(W_0)_x$ be its isotropy group. Then a smooth map $\varphi : \mathcal{P} \rightarrow A$ is called *W_0 -equivariant* if for every $x \in \mathcal{P}$ the point $a = \varphi(x)$ is a $(W_0)_x$ -fixed point and the induced homomorphism

$$\widehat{\varphi}_x : \widehat{\mathcal{C}}_{A,a} \rightarrow \widehat{\mathcal{C}}_{\mathfrak{a},x} \quad (7.2)$$

is $(W_0)_x$ -equivariant.

- d) Let $\Phi_0 \subseteq \Phi$ be a subroot system and let W_0 be its Weyl group. A smooth map $\varphi : \mathcal{P} \rightarrow A$ is called *Φ_0 -equivariant* if it is W_0 -equivariant and

$$\tilde{\alpha}(\varphi(x)) = 1 \quad (7.3)$$

for all $x \in \mathcal{P}$ and all roots $\alpha \in \Phi_0$ with $\alpha(x) = 0$.

We comment on these notions a bit more.

First of all, the notion of closedness can be rephrased in two ways. Since $\exp : \bar{\mathfrak{a}} \rightarrow A$ is a covering and \mathcal{P} is simply connected, the mapping φ can be lifted to a smooth map $\tilde{\varphi} : \mathcal{P} \rightarrow \bar{\mathfrak{a}}$. Because of the identification $\bar{\mathfrak{a}} \cong \bar{\mathfrak{a}}^*$ one can think of $\tilde{\varphi}$ as a 1-form. Then it is easy to see that φ is closed if and only if $\tilde{\varphi}$ is a closed 1-form (whence the name). Moreover, this shows that all closed maps are of the form $\exp(\nabla f)$ where f is a smooth function on \mathcal{P} and ∇f is its gradient.

For another way to see closedness, consider the derivative of φ at x which is an endomorphism

$$D_x \varphi : \bar{\mathfrak{a}} = T_x \mathfrak{a} \rightarrow T_{\varphi(x)} A = \bar{\mathfrak{a}}. \quad (7.4)$$

Then φ is closed if and only if $D_x \varphi$ is a self-adjoint operator for all $x \in \mathcal{P}$.

Since \mathcal{P} lies in \mathcal{A} , the fundamental alcove of W , every W_0 -orbit of \mathfrak{a} meets \mathcal{P} in at most one point. Therefore it can be shown (non-trivially!) that the condition of W_0 -equivariance is equivalent to the existence of a smooth (honestly) W_0 -equivariant map $\tilde{\varphi} : W_0 \mathcal{P} \rightarrow A$.

Finally, what is the difference between W_0 - and Φ_0 -equivariance? Actually not much. For $\alpha \in \Phi_0$ let $s_\alpha \in W_0$ be the corresponding reflection. Then $s_\alpha \in (W_0)_x$ if and only if $\alpha(x) = 0$. In this case, W_0 -equivariance implies $s_\alpha(a) = a$ where $a = \varphi(x)$. This means

$$\tilde{\alpha}^\vee(\tilde{\alpha}(a)) = 1_A \quad (7.5)$$

by equation (3.14). Applying $\tilde{\alpha}$ to both sides, we see (equation (3.13)) that

$$\tilde{\alpha}(\varphi(x))^2 = 1 \quad (7.6)$$

follows already from W_0 -equivariance. So Φ_0 -equivariance just means that $\tilde{\alpha}(\varphi(x))$ additionally equals 1 instead of -1 . Observe that applying instead of $\tilde{\alpha}$ a general character $\chi \in \Lambda$ to equation (7.5) we get

$$\tilde{\alpha}(\varphi(x))^{\langle \chi, \bar{\alpha}^\vee \rangle} = 1. \quad (7.7)$$

Thus, if the root α satisfies $\langle \Lambda, \bar{\alpha}^\vee \rangle = \mathbb{Z}$ then the condition (7.3) is in fact superfluous. This phenomenon will be explored more carefully in section 9.

Observe that all four conditions are local in \mathcal{P} . Hence the following makes sense:

7.1. Definition. For any open subset $U \subseteq \mathcal{P}$ let $\mathfrak{L}_{\mathcal{P},\Lambda}^\Phi(U)$ be the set of all smooth, closed, and Φ -equivariant maps $\varphi : U \rightarrow A$.

Clearly, $\mathfrak{L}_{\mathcal{P},\Lambda}^\Phi$ is a sheaf of abelian groups on \mathcal{P} . Let $x \in \mathcal{P}$. Then it is well known that the isotropy group W_x is the Weyl group of the (finite) root system

$$\Phi_x := \{\alpha \in \Phi \mid \alpha(x) = 0\}. \quad (7.8)$$

Moreover, W_y and Φ_y are contained in W_x and Φ_x , respectively, for y in a suitable open neighborhood U of x . This implies for the restriction to U :

$$\text{res}_U \mathfrak{L}_{\mathcal{P},\Lambda}^\Phi = \text{res}_U \mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi_x}. \quad (7.9)$$

This observation leads to the following generalization.

7.2. Definition. A *local system of roots* $\Phi(*)$ on \mathcal{P} is a family $(\Phi(x))_{x \in \mathcal{P}}$ of root systems on \mathfrak{a} such that for each $x \in \mathcal{P}$:

- a) $\Phi(y) = \Phi(x)_y$ for all y in a sufficiently small neighborhood of x in \mathcal{P} .
- b) Every root $\alpha \in \Phi(x)$ is either non-negative or non-positive on \mathcal{P} .

An *integral local system of roots* on \mathcal{P} is a pair $(\Phi(*), \Lambda)$ such that $(\Phi(x), \Lambda)$ is an integral root system for every $x \in \mathcal{P}$.

Observe that setting $y = x$ in condition a) yields that $\Phi(x)$ is centered at x , i.e., that $\alpha(x) = 0$ for all $\alpha \in \Phi(x)$. This implies in turn that all local root systems $\Phi(x)$ are finite. The second condition b) implies that the set $\Phi^+(x)$ of roots which are non-negative on \mathcal{P} forms a set of positive roots for $\Phi(x)$. This allows to define a set of simple roots

$$S(x) \subseteq \Phi^+(x) \subseteq \Phi(x). \quad (7.10)$$

An main (and often the only) example of an integral local system of roots on \mathcal{P} is $\Phi(x) := \Phi_x$ where (Φ, Λ) is an integral root system on \mathfrak{a} such that \mathcal{P} is entirely contained in an alcove of Φ . Such local systems are called *trivial*. For typical examples of trivial local root systems see figures (11.30), (11.34), and (11.38) where the gray area is \mathcal{P} and the dashed lines denote the reflection hyperplanes.

Now it is easy to extend the definition of $\mathfrak{L}_{\mathcal{P},\Lambda}^\Phi$.

7.3. Definition. Let $(\Phi(*), \Lambda)$ be an integral local system of roots on \mathcal{P} . Then $\mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi(*)}$ is the intersection of all $\mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi(x)}$, $x \in \mathcal{P}$, inside the sheaf of A -valued maps on \mathcal{P} . Concretely, given $U \subseteq \mathcal{P}$ open, then $\mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi(*)}(U)$ is the set of all smooth, closed maps $\varphi : U \rightarrow A$ which are $\Phi(x)$ -equivariant for all $x \in U$.

Observe that the coherence property a) implies that

$$\text{res}_U \mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi(*)} = \text{res}_U \mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi(x)}. \quad (7.11)$$

where U is a suitable open neighborhood of $x \in \mathcal{P}$.

Now let (M, m) be a multiplicity free quasi-Hamiltonian $K\tau$ -manifold. Let $m_+ : M \rightarrow \mathcal{P}_M$ be its invariant moment map. For any $U \subseteq \mathcal{P}_M$ open let $M(U) := m_+^{-1}(U)$. This

is again a multiplicity free quasi-Hamiltonian $K\tau$ -manifold. Let $\underline{\text{Aut}}_M$ be the sheaf of groups on \mathcal{P}_M defined by $\underline{\text{Aut}}_M(U) = \text{Aut}(M(U))$.

7.4. Theorem. *Let K be a simply connected compact Lie group and M be a convex multiplicity free quasi-Hamiltonian $K\tau$ -manifold with momentum image $\mathcal{P} = \mathcal{P}_M$ and character group $\Lambda = \Lambda_M$. Then there is an integral local system of roots $(\Phi(*), \Lambda)$ on \mathcal{P} such that*

$$\underline{\text{Aut}}_M \cong \mathfrak{L}_{\mathcal{P}, \Lambda}^{\Phi(*)}. \quad (7.12)$$

Proof. Fix $a \in \mathcal{P}$, let $u = \exp(x)$ be its image in K , and let L be the centralizer of u in K (with respect to the twisted action). Then Theorem 5.4 shows in particular that there exists an open neighborhood U of $x \in \mathcal{P}$ such that

$$\underline{\text{Aut}}_{M(U)} = a + \underline{\text{Aut}}_{M_0}. \quad (7.13)$$

Here, the left hand side denotes the sheaf of automorphisms of $M(U) = m_+^{-1}(U)$ as a quasi-Hamiltonian $K\tau$ -manifold. The right hand side is the sheaf of automorphisms of $M_0 = \log M_U$ as Hamiltonian L -manifold. The latter is a sheaf on $U - a$, so the “ $a+$ ” indicates translation back to U . Now the sheaf $\underline{\text{Aut}} M_0$ has been determined in [Kno11, Thm. 9.2] with the result that there is a unique finite root system $\Phi(x)$, centered at x , and an isomorphism

$$\mathfrak{L}_{U, \Lambda}^{\Phi(x)} \cong \underline{\text{Aut}}_{M_0}. \quad (7.14)$$

Thereby, the root systems $\Phi(x)$ indeed form a local systems of roots by [Kno11, eqn. (9.4)].

This already shows that $\underline{\text{Aut}}_M$ and $\mathfrak{L}_{\mathcal{P}, \Lambda}^{\Phi(*)}$ are locally isomorphic. It remains to show that the isomorphisms

$$\psi_U : \text{res}_U \mathfrak{L}_{\mathcal{P}, \Lambda}^{\Phi(*)} = \mathfrak{L}_{U, \Lambda}^{\Phi(x)} \xrightarrow{\sim} \text{res}_U \underline{\text{Aut}}_M \quad (7.15)$$

are compatible. For this we use that \mathcal{P}^0 , the interior of \mathcal{P} , is dense in \mathcal{P} . Thus, also $U^0 = \mathcal{P}^0 \cap U$ is dense in U . Therefore it suffices to show that $\text{res}_{U^0} \psi_U$ has a description which is independent of the choice of x and U .

To this end, observe that all roots $\alpha \in \Phi(x)$ are strictly positive on \mathcal{P}^0 and therefore also on U^0 . This implies that $\text{res}_{U^0} \mathfrak{L}_{U, \Lambda}^{\Phi(x)} = \mathfrak{L}_{U^0, \Lambda}^{\emptyset}$ is just the sheaf of smooth closed maps $\varphi : U^0 \rightarrow A$. As mentioned above, all such φ are of the form $\varphi = \exp \nabla f$ where f is a smooth function on U^0 . Since $(m_0)_+$ and m_+ are smooth over \mathcal{P}^0 , the functions $F_0 := f \circ (m_0)_+$ and $F = f \circ m_+$ are smooth on M_{U^0} and $M(U^0)$, respectively. Moreover, it follows from [Kno11, Thm. 9.1i)] that φ is the flow of the Hamiltonian vector field H_{F_0} at time $t = 1$. The compatibility of the Hamiltonian vector fields H_{F_0} and H_F (Lemma 5.5) shows that $\psi_U(\varphi)$ is the flow of H_F at $t = 1$. This shows that any two isomorphisms ψ_U and $\psi_{U'}$ coincide on $U \cap U' \cap \mathcal{P}^0$ and therefore, by density, also on $U \cap U'$. \square

8. The global Weyl group

In this and the following section we state and prove a criterion for when an integral local system of roots $(\Phi(*), \Lambda)$ on a set \mathcal{P} is trivial. For this consider first the system $(W(x))_{x \in \mathcal{P}}$ of Weyl groups of $\Phi(x)$. It forms a *local system of reflection groups* in the sense that for all $x \in \mathcal{P}$:

- a) $W(y) = W(x)_y$ for all y in a sufficiently small neighborhood of x in \mathcal{P} .

- b) Let $s \in W(x)$ be a reflection with fixed point set H_s . Then \mathcal{P} lies entirely in one of the two closed halfspaces of \mathfrak{a} which are defined by H_s .

Observe that again the case $y = x$ of condition [a\)](#) implies that x is a fixed point of $W(x)$. In particular, all local groups $W(x)$ are finite. In the following let $W_{\text{gl}} \subseteq M(\mathfrak{a})$ be the subgroup generated by the union of all $W(x)$, $x \in \mathcal{P}$. Our aim is to prove that $W(x) = (W_{\text{gl}})_x$ for all $x \in \mathcal{P}$.

To this end, we first state a mostly classical criterion for when a given set of reflections is the set of simple reflections for an Euclidean reflection group.

8.1. Lemma. *Let $\alpha_1, \dots, \alpha_n$ be non-constant affine linear functions on \mathfrak{a} with:*

- a) *For any $i \neq j$, the angle between $\bar{\alpha}_i$ and $\bar{\alpha}_j$ equals $\pi - \frac{\pi}{\ell}$ with $\ell \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.*
b) *There is a point $x \in \mathfrak{a}$ with $\alpha_i(x) > 0$ for all $i = 1, \dots, n$.*

Let $W \subseteq M(\mathfrak{a})$ be the group generated by the reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$. Then W is an Euclidean reflection group,

$$\mathcal{A} := \{x \in \mathfrak{a} \mid \alpha_1(x) \geq 0, \dots, \alpha_n(x) \geq 0\} \quad (8.1)$$

is an alcove for W , and the reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$ are precisely the simple reflection with respect to \mathcal{A} .

Proof. Condition [b\)](#) implies that \mathcal{A} is a convex polyhedron with non-empty interior. Let, after renumbering, $\alpha_1, \dots, \alpha_m$ be the non-redundant functions defining \mathcal{A} , i.e., whose intersection $\{\alpha_i = 0\} \cap \mathcal{A}$ is of codimension 1 in \mathcal{A} . Then a classical theorem (see e.g. [\[Vin71, Thm. 1\]](#) for a much more general statement) asserts that, under condition [a\)](#), $s_{\alpha_1}, \dots, s_{\alpha_m}$ are the simple reflections for an Euclidean reflection group W and that \mathcal{A} is a fundamental domain. So, it remains to show that $m = n$. Suppose not. Then α_{m+1} would be redundant. This implies that there are real numbers $c_1, \dots, c_m \geq 0$ such that $\alpha_{m+1} = \sum_{i=1}^m c_i \alpha_i$. From [a\)](#) we get that

$$\langle \bar{\alpha}_i, \bar{\alpha}_{m+1} \rangle = \|\bar{\alpha}_i\| \|\bar{\alpha}_{m+1}\| \cdot \cos(\pi - \frac{\pi}{\ell}) \leq 0 \quad (8.2)$$

for $i = 1, \dots, m$ and therefore the contradiction $\langle \bar{\alpha}_{m+1}, \bar{\alpha}_{m+1} \rangle \leq 0$. \square

Here is our criterion:

8.2. Proposition. *Let $\mathcal{P} \subseteq \mathfrak{a}$ be a convex subset with non-empty interior and let $W(*)$ be a local system of reflection groups on \mathcal{P} . Let $W_{\text{gl}} \subseteq M(\mathfrak{a})$ be the group generated by the union of all $W(x)$, $x \in \mathcal{P}$. Assume moreover that every W_{gl} -orbit meets \mathcal{P} in at most one point. Then W_{gl} is an Euclidean reflection group with $W(x) = (W_{\text{gl}})_x$ for all $x \in \mathcal{P}$. Moreover, there is a unique alcove \mathcal{A} of W_{gl} with $\mathcal{P} \subseteq \mathcal{A}$ and \mathcal{P} has a non-empty intersection with every wall of \mathcal{A} .*

Proof. First we claim that

$$W(x)_y = W(y)_x \quad \text{for all } x, y \in \mathcal{P}. \quad (8.3)$$

Indeed, let $l = [x, y] \subseteq \mathfrak{a}$ be the line segment joining x and y . Then $l \subseteq \mathcal{P}$ since \mathcal{P} is convex. For any $z \in l$ let

$$W(z)_l := \{w \in W(z) \mid wu = u \text{ for all } u \in l\} \quad (8.4)$$

Then

$$W(u)_l = (W(z)_u)_l = W(z)_l \quad (8.5)$$

for all $u \in l$ which are sufficiently close to z . This means that the map $z \mapsto W(z)_l$ is locally constant, hence constant, on l . Thus

$$W(x)_y = W(x)_l = W(y)_l = W(y)_x \quad (8.6)$$

finishing the proof of the claim.

Let $s = s_\alpha \in W_{\text{gl}}$ be a reflection with fixed point set $H := \{\alpha = 0\}$. We claim that H does not meet \mathcal{P}^0 , the open interior of \mathcal{P} . Otherwise, there would be points $x, y \in \mathcal{P}^0$ with $\alpha(x) > 0$ and $\alpha(y) < 0$. The line segment joining x and y lies entirely in \mathcal{P}^0 and meets H in exactly one point z . Moreover there is $\varepsilon > 0$ such that both points $z_\pm := z \pm \varepsilon \bar{\alpha}$ are in \mathcal{P}^0 . But then z_+ and $z_- = s(z_+)$ would be two different points of \mathcal{P} lying in the same W_{gl} -orbit contradicting our assumption.

The claim implies that \mathcal{P}^0 , being connected, lies entirely in one of the open halfspaces determined by H . Hence \mathcal{P} , its closure, lies entirely in one of the two closed halfspaces determined by H .

This reasoning applies, in particular, to all reflections contained in $W(x)$. Thus, \mathcal{P} is contained in a unique Weyl chamber $C(x) \subseteq \mathfrak{a}$ for $W(x)$. This chamber determines in turn a set $S(x) \subset W(x)$ of simple reflections. It is well-known that for any $y \in C(x)$ the set $S(x)_y := \{s \in S(x) \mid sy = y\}$ is a set of simple reflections for $W(x)_y$. Therefore equation (8.3) implies that

$$S(x)_y = S(y)_x \quad \text{for all } x, y \in \mathcal{P}. \quad (8.7)$$

Now let S be the union of all $S(x)$, $x \in \mathcal{P}$. Then

$$S(x) = \{s \in S \mid sx = x\} \quad (8.8)$$

for all $x \in \mathcal{P}$. Indeed, let $s \in S$ with $sx = x$. Then $s \in S(y)$ for some $y \in \mathcal{P}$. Thus, $s \in S(y)_x = S(x)_y \subseteq S(x)$.

For each $s \in S$ choose affine linear functions α_s with $s = s_{\alpha_s}$ and such that $\alpha_s \geq 0$ on \mathcal{P} . We are going to show that $\{\alpha_s \mid s \in S\}$ satisfies the assumptions of Lemma 8.1.

Let $s_1 \neq s_2 \in S$. Put $\alpha_i := \alpha_{s_i}$ and $H_i := \{\alpha_i = 0\}$. Assume first that H_1 and H_2 are parallel. Then $\bar{\alpha}_1 = c\bar{\alpha}_2$ with $c \neq 0$ and we have to show that $c < 0$. The functions α_i vanish, by construction, at some points $x_i \in \mathcal{P}$. Put $t := x_1 - x_2 \in \bar{\mathfrak{a}}$. Then $\langle \bar{\alpha}_1, t \rangle = -\alpha_1(x_2) < 0$ and $\langle \bar{\alpha}_2, t \rangle = \alpha_2(x_1) > 0$ which shows $c < 0$.

Now assume that H_1 and H_2 are not parallel. Then $E := H_1 \cap H_2$ is a subspace of codimension two. Let $W' \subseteq W$ be the dihedral group generated by s_1 and s_2 and let θ be the angle between $\bar{\alpha}_1$ and $\bar{\alpha}_2$. Then W' contains the rotation r around E with angle 2θ . If r had infinite order then the union of all $\langle r \rangle$ -translates of, say, H_1 would be dense in \mathfrak{a} . But that contradicts the assumption that every W_{gl} -orbit meets \mathcal{P} at most once. Therefore W' is a finite reflection group.

Now we claim that $\{s_1, s_2\}$ forms a set of simple reflections for W' . If $E \cap \mathcal{P} \neq \emptyset$ this is clear since then $s_1, s_2 \in S(x)$ for all $x \in E \cap \mathcal{P}$ (by eqn. (8.8)). So assume $E \cap \mathcal{P} = \emptyset$. Let C' be the unique Weyl chamber of W' which contains \mathcal{P} and let $s'_i \in W'$, $i = 1, 2$, be

the corresponding simple reflections. Choose functions α'_i with $s'_i = s_{\alpha'_i}$ such that $\alpha'_i \geq 0$ on \mathcal{P} . Observe that

$$E = \{\alpha_1 = \alpha_2 = 0\} = \{\alpha'_1 = \alpha'_2 = 0\} = \mathfrak{a}^{W'}. \quad (8.9)$$

Now fix $i \in \{1, 2\}$. Then $\alpha_i = c_1\alpha'_1 + c_2\alpha'_2$ for some real numbers $c_1, c_2 \geq 0$. Suppose $c_1, c_2 > 0$, i.e., s_i is not simple. By construction $s_i \in W(x)$ for some $x \in \mathcal{P}$. Then

$$0 = \alpha_i(x) = c_1\alpha'_1(x) + c_2\alpha'_2(x) \quad (8.10)$$

implies $\alpha'_1(x) = \alpha'_2(x) = 0$ and therefore $x \in \mathcal{P} \cap E$ which is excluded.

The fact that s_1 and s_2 are simple reflections of W' implies that the angle of $\bar{\alpha}_1$ and $\bar{\alpha}_2$ is of the form $\pi - \frac{\pi}{\ell}$ with $\ell \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Since condition *b)* is obvious from $\mathcal{P}^0 \subseteq \mathcal{A}$ we can apply Lemma 8.1 and infer that W_{gl} is an Euclidean reflection group with alcove $\mathcal{A} \supseteq \mathcal{P}$ and that S is a set of simple reflections of W . Finally, (8.8) implies

$$W_x = \langle s \in S \mid sx = x \rangle = \langle S(x) \rangle = W(x). \quad (8.11)$$

for all $x \in \mathcal{P}$. The last assertion holds by construction. \square

Applying this to multiplicity free manifolds we get:

8.3. Corollary. *Let $\Phi(*)$ be the local system of roots of a convex multiplicity free quasi-Hamiltonian $K\tau$ -manifold (as in Theorem 7.4). Let $W(*)$ be the corresponding local system of reflection groups. Then there is an Euclidean reflection group $W_M \subseteq M(\mathfrak{a}_M)$ such that $W(x) = (W_M)_x$ for all $x \in \mathcal{P}_M$.*

Proof. This follows from Proposition 8.2 as soon as we have shown that each W_{gl} -orbit meets \mathcal{P}_M in at most one point. To this end recall that the Weyl group W_τ of the root system Φ_τ (see Theorem 4.2) acts on the Cartan subspace \mathfrak{a} as an Euclidean reflection group. Let N and C be the normalizer and centralizer, respectively, of $\mathfrak{a}_M \subseteq \mathfrak{a}$. Then $\tilde{W} := N/C$ acts properly on \mathfrak{a}_M and each \tilde{W} -orbit meets $\mathfrak{a}_M \cap \mathcal{P}$ at most once. Now the claim follows from the fact that each local Weyl group $W(x)$, hence W_{gl} , is a subgroup of \tilde{W} (see, e.g., [Kno11, Thm. 3.2]). \square

9. The global root system

In this section, we complete the proof that the local system of roots attached to a multiplicity free manifold is trivial. We already know that there is a global Weyl group and a weight lattice. So for the root system there are only finitely many possibilities which we are first going to investigate.

More abstractly, let $W \subseteq M(\mathfrak{a})$ be an Euclidean reflection group, $\Lambda \subseteq \bar{\mathfrak{a}}$ a weight lattice for W (see Definition 3.3), and $\mathcal{A} \subseteq \mathfrak{a}$ an alcove. Let $S \subseteq W$ be the set of simple reflections with respect to \mathcal{A} .

For $s \in S$ let

$$\Lambda^{\pm s} := \{\chi \in \Lambda \mid s(\chi) = \pm\chi\}. \quad (9.1)$$

Since $\Lambda^{-s} \cong \mathbb{Z}$ there is a unique affine function α_s^{prim} on \mathfrak{a} with $\mathfrak{a}^{(s)} = \{\alpha_s^{\text{prim}} = 0\}$, which is non-negative on \mathcal{A} and such that $\bar{\alpha}_s^{\text{prim}}$ is a generator of Λ^{-s} . If α_s is a root for s in some root system then necessarily $\alpha_s = \alpha_s^{\text{prim}}$ or $\alpha_s = 2\alpha_s^{\text{prim}}$. In the second case $\bar{\alpha}_s^\vee = \frac{1}{2}(\bar{\alpha}_s^{\text{prim}})^\vee$ which therefore can only occur if $\langle \Lambda, (\bar{\alpha}_s^{\text{prim}})^\vee \rangle = 2\mathbb{Z}$. There is another

way to put this: $\langle \bar{\alpha}_s^{\text{prim}}, \bar{\alpha}_s^\vee \rangle = 1$ implies that Λ^{-s} is an $\langle s \rangle$ -equivariant direct summand of Λ .

9.1. Definition. *a)* A simple reflection $s \in S$ is called *ambiguous* if $\langle \Lambda, (\bar{\alpha}_s^{\text{prim}})^\vee \rangle = 2\mathbb{Z}$ or, equivalently, if $\Lambda = \Lambda^{+s} \oplus \Lambda^{-s}$. The set of ambiguous simple reflections is denoted by $S_{\text{amb}} \subseteq S$.

b) Let (Φ, Λ) be an integral root system with Weyl group $W_\Phi = W$. Then

$$S_{\text{amb}}(\Phi) := \{s \in S \mid \alpha_s = 2\alpha_s^{\text{prim}}\} \subseteq S_{\text{amb}}. \quad (9.2)$$

Remarks. *a)* The notion of ambiguity depends on the lattice W . Let, e.g., W be a Weyl group of type B_n and let s be the reflection corresponding to the short simple root. Then s is ambiguous with respect to the roots lattice but not with respect to the weight lattice.

b) Some simple roots are never ambiguous. Assume for example that there is a simple root $s' \in S$ with $(ss')^3 = 1$. This means that s and s' are joined by a simple edge in the Coxeter diagram of W . Then

$$\langle \bar{\alpha}_{s'}^{\text{prim}}, (\bar{\alpha}_s^{\text{prim}})^\vee \rangle = -1 \quad (9.3)$$

implies that s is not ambiguous.

Now we classify all root systems Φ with a given weight lattice Λ and Weyl group W .

9.2. Lemma. *Let $W \subseteq M(\mathfrak{a})$ be an Euclidean reflection group, let $\mathcal{A} \subseteq \mathfrak{a}$ be an alcove of W , let $S \subseteq W$ be the corresponding set of simple reflections, and let $\Lambda \subseteq \bar{\mathfrak{a}}$ be a weight lattice. Then:*

a) No two distinct elements of S_{amb} are conjugate within W .

b) The map $\Phi \mapsto S_{\text{amb}}(\Phi) \subseteq S_{\text{amb}}$ is a bijection between root systems Φ with Weyl group W and weight lattice Λ , and subsets of S_{amb} .

Proof. *a)* It is well-known (see e.g. [Bou68] IV, §1, Prop. 3) that two simple reflections s, s' in a Coxeter group are conjugate if and only if there is a string of simple reflections $s = s_1, s_2, \dots, s_n = s'$ such that the order of $s_i s_{i+1}$ is odd for all $i = 1, \dots, n-1$. For Weyl groups this happens only if the order is 3. But then, by the Remark *b)* above, neither s nor s' is ambiguous.

b) We construct the inverse mapping. For $I \subseteq S_{\text{amb}}$ let

$$\alpha_s^I := \begin{cases} 2\alpha_s^{\text{prim}} & \text{if } s \in I \\ \alpha_s^{\text{prim}} & \text{if } s \in S \setminus I \end{cases} \quad (9.4)$$

and $S_I := \{\alpha_s^I \mid s \in S\}$. Then part *a)* shows that $\Phi_I = WS_I$ is a *reduced* root system with simple roots S_I and $S_{\text{amb}}(\Phi_I) = I$. \square

Next, we determine the set S_{amb} of ambiguous roots.

9.3. Proposition. *Let W , \mathcal{A} , S , and Λ be as in Lemma 9.2.*

a) *Reduction to irreducible reflection groups:* Let $S_0 \subseteq S$ be a connected component of the Coxeter diagram of S containing an ambiguous root. Let $\bar{\alpha}_0 := \langle \bar{\alpha}_s^{\text{prim}} \in \bar{\mathfrak{a}} \mid s \in S_0 \rangle_{\mathbb{R}}$ and $\Lambda_0 := \Lambda \cap \bar{\alpha}_0$. Then Λ_0 is a W -equivariant direct summand of Λ .

- b) Assume (W, S) to be irreducible with $S_{\text{amb}} \neq \emptyset$. Then $\Lambda = \langle \bar{\alpha}_s^{\text{prim}} \mid s \in S \rangle_{\mathbb{Z}}$. In particular, (W, Λ) is determined by the root system Φ_{\emptyset} (see proof of Lemma 9.2 for the notation).
- c) The irreducible root systems which are of the form Φ_{\emptyset} are listed in the following table. The set S_{amb} is marked by asterisks.

Φ_{\emptyset}	Diagram
A_1	\bullet^*
$B_n \quad (n \geq 2)$	$\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet \Rightarrow \bullet^*$
$A_1^{(1)}$	$\bullet^* \Leftrightarrow \bullet^*$
$B_2^{(1)}$	$\bullet \Leftrightarrow \bullet^* \Leftrightarrow \bullet$
$B_n^{(1)} \quad (n \geq 3)$	$\begin{array}{c} \bullet \\ \bullet \end{array} \text{---} \bullet \text{---} \dots \text{---} \bullet \Rightarrow \bullet^*$
$D_{n+1}^{(2)} \quad (n \geq 2)$	$\bullet^* \Leftrightarrow \bullet \text{---} \dots \text{---} \bullet \Rightarrow \bullet^*$

Proof. We first prove b). Assume therefore that (W, S) is irreducible and that $\Phi = \Phi_{\emptyset}$ for some choice of Λ . Let $s \in S_{\text{amb}}$. Then $\langle \beta, \bar{\alpha}_s^{\vee} \rangle$ must be even for all $\beta \in S$. The classification of (affine) root systems (see e.g. [Kac90] or [Mac03]) shows that the Dynkin diagram of Φ is one of the items in the table above or is one of the following

$$\begin{array}{c}
 A_2^{(2)} \\
 A_{2n}^{(2)} \quad (n \geq 2)
 \end{array}
 \left|
 \begin{array}{c}
 \beta \Rightarrow \bullet^* \\
 \beta \Rightarrow \bullet \text{---} \dots \text{---} \bullet \Rightarrow \bullet^*
 \end{array}
 \right.
 \quad (9.5)$$

Moreover, s corresponds to one of vertices marked by an asterisk. Now from $\bar{\alpha}_s \in \Lambda$ and $\gamma := \frac{1}{2}\bar{\alpha}_s^{\vee} \in \Lambda^{\vee}$ we get

$$\langle W\bar{\alpha}_s \rangle \subseteq \Lambda \subseteq \langle W\gamma \rangle^{\vee}. \quad (9.6)$$

It is easy to verify case-by-case that $W\bar{\alpha}_s$ equals either $\{\varepsilon_1, \dots, \varepsilon_n\}$ or $\{\pm\varepsilon_1, \dots, \pm\varepsilon_n\}$ where $\varepsilon_1, \dots, \varepsilon_n$ is an orthogonal basis of $\bar{\mathfrak{a}}$ all whose elements have the same length. This and $\langle \bar{\alpha}_s, \gamma \rangle = 1$ immediately imply that the outer lattices of (9.6) coincide. Hence, Λ equals the root lattice of Φ proving b).

From this also a) follows: Let $\tilde{\Lambda}_0$ be the orthogonal projection of Λ to $\bar{\alpha}_0$. Then the inclusions 9.6 hold for both Λ_0 and $\tilde{\Lambda}_0$. So $\Lambda_0 = \tilde{\Lambda}_0$ which implies that Λ_0 is a W -direct summand of Λ .

Finally, for c), observe that the root systems $A_{2n}^{(2)}$ ($n \geq 1$) cannot be of the form Φ_{\emptyset} since $s_{\beta} \in S_{\text{amb}}(\Phi)$. \square

Next, we prove a refinement of Proposition 8.2 to root systems.

9.4. Proposition. *Let $\mathcal{P} \subseteq \mathfrak{a}$ be a convex subset with non-empty interior and let $(\Phi(*), \Lambda)$ be an integral local system of roots on \mathcal{P} . Let W be the group generated by all the local Weyl groups $W(x)$, $x \in \mathcal{P}$, and assume that every W -orbit meets \mathcal{P} at most once. Then there is an integral root system (Φ, Λ) with $\Phi(x) = \Phi_x$ for all $x \in \mathcal{P}$.*

Proof. Proposition 8.2 implies that W is an Euclidean reflection group with $W(x) = W_x$ for all $x \in \mathcal{P}$. In particular, $(\Phi(x), \Lambda)$ is an integral root system with Weyl group W_x . Let $S \subseteq W$ be the set of simple reflections. Then $S_x = S \cap W_x$ is a set of simple roots for $\Phi(x)$. Moreover, $\Phi(x)$ is a root system with Weyl group W_x such that Λ is a weight lattice and

$$S_{\text{amb}}(x) := \{s \in S_x \mid \bar{\alpha}_s^{\text{prim}} \notin \Phi(x)\} \subseteq S_{\text{amb}} \quad (9.7)$$

Now the same argument as for (8.3) also shows

$$\Phi(x)_y = \Phi(y)_x \text{ for all } x, y \in \mathcal{P}. \quad (9.8)$$

This implies that whenever $s \in S_x \cap S_y$ then $s \in S_{\text{amb}}(x)$ if and only if $S_{\text{amb}}(y)$. Thus, the union $S_0 = \bigcup_{x \in \mathcal{P}} S_{\text{amb}}(x)$ has the property that $S_0 \cap S_x = S_{\text{amb}}(x)$ for all x . Therefore the root system Φ attached to $S_0 \subseteq S_{\text{amb}}$ satisfies $\Phi_x = \Phi(x)$, as required. \square

Now we can improve on Theorem 7.4:

9.5. Theorem. *Let K be a simply connected compact Lie group with twist τ and M be a convex multiplicity free quasi-Hamiltonian $K\tau$ -manifold. Then there is a unique affine root system Φ_M on \mathfrak{a}_M such that*

- Λ_M is a weight lattice for Φ_M .
- \mathcal{P}_M is contained in a (unique) alcove \mathcal{A} of Φ_M .
- \mathcal{P}_M intersects every wall of \mathcal{A} .
- The sheaf of automorphisms $\underline{\text{Aut}}_M$ is canonically isomorphic to $\mathfrak{L}_{\mathcal{P}_M, \Lambda_M}^{\Phi_M}$.

Proof. Apply Proposition 9.4 to Theorem 9.5. The condition on W has been verified in the proof of Corollary 8.3. \square

10. Cohomology computations

In this section, we provide the last step of the proof of our main classification Theorem 6.7.

10.1. Theorem. *Let (Φ, Λ) be an integral root system on the Euclidean affine space \mathfrak{a} , let \mathcal{A} be a fixed alcove of Φ , and let $\mathcal{P} \subseteq \mathcal{A}$ be a locally closed convex subset with non-empty interior. Then $H^i(\mathcal{P}, \mathfrak{L}_{\mathcal{P}, \Lambda}^{\Phi}) = 0$ for all $i \geq 1$.*

The proof will occupy the rest of this section. We start with a reduction step:

10.2. Lemma. *Let $\Lambda_1, \Lambda_2 \subseteq \bar{\mathfrak{a}}$ be two commensurable weight lattices for Φ . Then $H^i(\mathcal{P}, \mathfrak{L}_{\mathcal{P}, \Lambda_1}^{\Phi}) = H^i(\mathcal{P}, \mathfrak{L}_{\mathcal{P}, \Lambda_2}^{\Phi})$ for all $i \geq 1$.*

Proof. By replacing Λ_1 with the intersection $\Lambda_1 \cap \Lambda_2$ we may assume $\Lambda_1 \subseteq \Lambda_2$. Then $A_1 := \bar{\mathfrak{a}}/\Lambda_1^{\vee}$ is a quotient of $A_2 := \bar{\mathfrak{a}}/\Lambda_2^{\vee}$ with kernel

$$E := \Lambda_1^{\vee}/\Lambda_2^{\vee} \subseteq A_2^{\Phi}. \quad (10.1)$$

Let $U \subseteq \mathcal{P}$ be convex and open. Then any map $\varphi_1 : U \rightarrow A_1$ can be lifted to a map $\varphi_2 : \mathcal{P} \rightarrow A_2$. Moreover, φ_2 is smooth, closed, and Φ -invariant if and only if φ_1 is. Thus, we get a short exact sequence of sheaves

$$0 \longrightarrow E_{\mathcal{P}} \longrightarrow \mathfrak{L}_{\mathcal{P}, \Lambda_2}^{\Phi} \longrightarrow \mathfrak{L}_{\mathcal{P}, \Lambda_1}^{\Phi} \longrightarrow 0 \quad (10.2)$$

where $E_{\mathcal{P}}$ denotes the constant sheaf on \mathcal{P} with fiber E . Since \mathcal{P} is convex, we have $H^i(\mathcal{P}, E_{\mathcal{P}}) = 0$ for $i \geq 1$. From this the assertion follows. \square

A weight lattice will be called *of adjoint type* if

$$\Lambda = \mathbb{Z}\overline{\Phi} \oplus \Lambda^W \subseteq \mathbb{R}\overline{\Phi} \oplus \overline{\mathfrak{a}}^W = \overline{\mathfrak{a}}. \quad (10.3)$$

Since every weight lattice Λ is commensurable to $\mathbb{Z}\overline{\Phi} \oplus \Lambda^W$ the Lemma allows us to assume that Λ is of adjoint type.

Now recall that the sections of $\mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi}$ are the smooth, closed, Φ -equivariant maps $\varphi : U \rightarrow A$ where A is the torus $\overline{\mathfrak{a}}/\Lambda^{\vee}$ and $U \subseteq \mathcal{P}$ is open. To construct maps of this type, consider a smooth function f defined on U . As explained in section 7, the map

$$\varepsilon(f) : U \rightarrow A : x \mapsto \exp(2\pi\nabla f(x)) \quad (10.4)$$

is smooth and closed. It is Φ -equivariant whenever f is W -invariant in the sense that for each $x \in \mathcal{P}$ the Taylor series \widehat{f} of f in x is W_x -invariant. Let $\mathcal{C}_{\mathcal{P}}^W$ be the sheaf of W -invariant smooth functions on \mathcal{P} . This way, we get a homomorphism of sheaves

$$\varepsilon : \mathcal{C}_{\mathcal{P}}^W \rightarrow \mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi}. \quad (10.5)$$

Our first goal is to study the cokernel of this map. To this end, consider the subgroup

$$A^{\Phi} := \{u \in A \mid \tilde{\alpha}(u) = 1 \text{ for all } \alpha \in \Phi\}. \quad (10.6)$$

Its elements are called the Φ -fixed points of A . By (3.14), they form a subgroup of A^W , the group of W -fixed points. Of particular interest will be the component group $\pi_0(A^{\Phi_x})$.

One can localize these constructions. For any $x \in \mathcal{P}$ consider the groups A^{Φ_x} and $\pi_0(A^{\Phi_x})$. If y is close to x then $\Phi_y \subseteq \Phi_x$ and therefore

$$A^{\Phi_x} \subseteq A^{\Phi_y}. \quad (10.7)$$

This shows that there is a constructible sheaf $\mathfrak{C}_{\mathcal{P}}$ such that $\pi_0(A^{\Phi_x})$ is the stalk at x and the restriction maps $\pi_0(A^{\Phi_x}) \rightarrow \pi_0(A^{\Phi_y})$ are induced by (10.7). Its significance is given by

10.3. Lemma. *There is an exact sequence*

$$\mathcal{C}_{\mathcal{P}}^W \xrightarrow{\varepsilon} \mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi} \xrightarrow{\eta} \mathfrak{C}_{\mathcal{P}} \rightarrow 0. \quad (10.8)$$

Proof. Let $U \subseteq \mathcal{P}$ be open and fix $x \in U$. If $\varphi \in \mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi}(U)$ then $\varphi(x) \in A^{\Phi_x}$, by Φ -equivariance. Thus we can define $\eta(\varphi)(x)$ to be the image of $\varphi(x)$ in $\pi_0(A^{\Phi_x})$.

Now let $u \in A^{\Phi_x}$ be a representative of some element $\overline{u} \in \pi_0(A^{\Phi_x})$. Then the constant map $\varphi : x \mapsto u$ is a section of $\mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi}$ with $\eta(\varphi) = \overline{u}$. This shows that η is surjective.

On the other hand, for any section f of $\mathcal{C}_{\mathcal{P}}^W$, the image $\varepsilon(f)(x)$ lies in $\exp(\overline{\mathfrak{a}}^{W_x}) = (A^{\Phi_x})^0$. This shows $\text{im } \varepsilon \subseteq \ker \eta$.

To show equality, let $\varphi : U \rightarrow A$ be a section of $\mathfrak{L}_{\mathcal{P},\Lambda}^{\Phi}$ with $\eta(\varphi) = 0$, i.e., $\varphi(x) \in (A^{\Phi_x})^0$ for all x . Then there is a lift $\tilde{\varphi} : U \rightarrow \overline{\mathfrak{a}}$ of φ with $\tilde{\varphi}(x) \in \overline{\mathfrak{a}}^{W_x}$. The last condition implies that $\tilde{\varphi}$ is W_x -equivariant. Since $\tilde{\varphi}$ is smooth and closed there is a smooth function f on U with $\nabla f = \tilde{\varphi}$. Let \widehat{f} be the Taylor series of f in x . Then the W_x -equivariance of $\tilde{\varphi}$ implies $\nabla^w \widehat{f} = \nabla \widehat{f}$ for all $w \in W_x$. Hence $c_w := \widehat{f} - \nabla^w \widehat{f}$ is a constant and $w \mapsto c_w$ is a homomorphism. We conclude $c_w = 0$, i.e., f is in fact W_x -invariant. Therefore, $\varphi = \varepsilon(f)$ is indeed in the image of ε . \square

To investigate the cohomology of $\mathfrak{C}_{\mathcal{P}}$ we need a more explicit description. The character group of A^{Φ} is given by

$$\Xi(A^{\Phi}) = \Lambda / \mathbb{Z}\overline{\Phi}. \quad (10.9)$$

In particular, $\pi_0(A^{\Phi}) = 0$ if and only if the root lattice $\mathbb{Z}\overline{\Phi}$ is a direct summand of Λ . More generally, we have

$$\Xi(\pi_0(A^{\Phi})) = \text{Tors}(\Lambda / \mathbb{Z}\overline{\Phi}) = \frac{\Lambda \cap \mathbb{R}\overline{\Phi}}{\mathbb{Z}\overline{\Phi}}. \quad (10.10)$$

Dualizing, this is equivalent to

$$\pi_0(A^{\Phi}) = \frac{(\mathbb{Z}\overline{\Phi})^{\vee}}{\Lambda^{\vee} + (\mathbb{R}\overline{\Phi})^{\vee}} \quad (10.11)$$

where $(\mathbb{Z}\overline{\Phi})^{\vee}$ is the coweight lattice and $(\mathbb{R}\overline{\Phi})^{\vee}$ is the orthogonal complement of $\mathbb{R}\overline{\Phi}$.

We compute $\mathfrak{C}_{\mathcal{P}}$ in two stages, the first being the case of finite root systems.

10.4. Lemma. *Assume Φ is finite and $\Lambda = \mathbb{Z}\overline{\Phi}$. Then $\mathfrak{C}_{\mathcal{P}} = 0$.*

Proof. Let $S = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots of $\overline{\Phi}$. Since these form a basis of $\overline{\mathfrak{a}}$ we get an isomorphism

$$\alpha_* : \overline{\mathfrak{a}} \rightarrow \mathbb{R}^n : x \mapsto (\langle \alpha_1, x \rangle, \dots, \langle \alpha_n, x \rangle) \quad (10.12)$$

For any subset $I \subseteq \{1, \dots, n\}$ let I' be its complement. Moreover, for $k \in \{\mathbb{R}, \mathbb{Z}\}$ we put

$$k^I := \{(x_i) \in \mathbb{R}^n \mid x_i \in k \text{ for } i \in I \text{ and } x_i = 0 \text{ otherwise}\} \cong k^{|I|} \quad (10.13)$$

For any $x \in \mathcal{P}$ let $I := \{i \mid \alpha_i(x) = 0\}$. Then α_* maps $(\mathbb{Z}\overline{\Phi}_x)^{\vee}$, $(\mathbb{R}\overline{\Phi}_x)^{\perp}$, and Λ^{\vee} to $\mathbb{Z}^I \oplus \mathbb{R}^{I'}$, $\mathbb{R}^{I'}$, and \mathbb{Z}^n , respectively. Now the claim follows from (10.11). \square

Now suppose Φ is an infinite irreducible root system with simple roots $S = \{\alpha_1, \dots, \alpha_n\}$. The *labels* of S are defined as the components of the unique primitive vector $(a_1, \dots, a_n) \in \mathbb{Z}_{>0}^n$ such that

$$a_1\overline{\alpha}_1 + \dots + a_n\overline{\alpha}_n = 0. \quad (10.14)$$

For $I \subsetneq \{1, \dots, n\}$ we define $I' \neq \emptyset$ as its complement and

$$d_I := \gcd\{a_j \mid j \in I'\}. \quad (10.15)$$

10.5. Lemma. *Assume Φ is an infinite irreducible root system and $\Lambda = \mathbb{Z}\overline{\Phi}$. For any $x \in \mathcal{A}$ let $I := \{i \mid \alpha_i(x) = 0\}$. Then there is a canonical isomorphism*

$$\pi_0(A^{\Phi_x}) \rightarrow \mathbb{Z}/d_I\mathbb{Z}. \quad (10.16)$$

Moreover, this isomorphism is compatible with the restriction homomorphisms of $\mathfrak{C}_{\mathcal{P}}$.

Proof. We keep the notation of the proof of Lemma 10.4. Let $\delta := (a_1, \dots, a_n) \in \mathbb{R}^n$ be the vector of labels. Then the map α_* of (10.12) identifies $\overline{\mathfrak{a}}$ with the hyperplane H of \mathbb{R}^n which is perpendicular to δ . Thus (10.11) becomes

$$\pi_0(A^{\Phi_x}) = \frac{(\mathbb{Z}^I \oplus \mathbb{R}^{I'}) \cap H}{(\mathbb{R}^{I'} \cap H) + (\mathbb{Z}^n \cap H)} \quad (10.17)$$

Now consider the homomorphism

$$p_I : (\mathbb{Z}^I \oplus \mathbb{R}^{I'}) \cap H \rightarrow \mathbb{Z}/d_I\mathbb{Z} : (x_i) \mapsto \sum_{i \in I} a_i x_i + d_I \mathbb{Z}. \quad (10.18)$$

Since $d_{I'}$ and d_I are coprime there are $a', a \in \mathbb{Z}$ with $a'd_{I'} + ad_I = 1$. Because $I' \neq \emptyset$, there is $(x_i) \in (\mathbb{Z}^I \oplus \mathbb{R}^{I'}) \cap H$ with $\sum_{i \in I} a_i x_i = a'd_{I'}$. Then $p_I(x_i) = 1$, i.e., p_I is onto.

Next we claim that the kernel of p_I is precisely $E := (\mathbb{R}^{I'} \cap H) + (\mathbb{Z}^n \cap H)$. Clearly $\mathbb{R}^{I'} \cap H \subseteq \ker p_I$. Let $(x_i) \in \mathbb{Z}^n \cap H$. Then

$$\sum_{i \in I} a_i x_i = - \sum_{j \in I'} a_j x_j \in d_I \mathbb{Z} \quad (10.19)$$

shows that $E \subseteq \ker p_I$. To show the converse, let $(x_i) \in \ker p_I$. Then, by definition, $\sum_{i \in I} a_i x_i \in d_I \mathbb{Z}$. Hence there is $(y_i) \in \mathbb{Z}^{I'}$ with

$$\sum_{i \in I} a_i x_i = - \sum_{j \in I'} a_j y_j. \quad (10.20)$$

Now define

$$\bar{x}_i := \begin{cases} x_i & \text{if } i \in I \\ y_i & \text{if } i \in I' \end{cases} \quad (10.21)$$

Then $(\bar{x}_i) \in \mathbb{Z}^n \cap H$ with $(x_i) - (\bar{x}_i) \in \mathbb{R}^{I'} \cap H$, proving the claim. Thus p_I induces an isomorphism between $\pi_0(A^{\Phi_x})$ and $\mathbb{Z}/d_I \mathbb{Z}$.

For the final claim, we denote I by I_x . Let $y \in \mathcal{P}$ be close to x . Then $I_y \subseteq I_x$ and therefore $d_{I_y} | d_{I_x}$. Thus, we have to show that the diagram

$$\begin{array}{ccc} (\mathbb{Z}^{I_x} \oplus \mathbb{R}^{I'_x}) \cap H & \xrightarrow{p_{I_x}} & \mathbb{Z}/d_{I_x} \mathbb{Z} \\ \downarrow & & \downarrow [1] \mapsto [1] \\ (\mathbb{Z}^{I_y} \oplus \mathbb{R}^{I'_y}) \cap H & \xrightarrow{p_{I_y}} & \mathbb{Z}/d_{I_y} \mathbb{Z} \end{array} \quad (10.22)$$

commutes. But this follows from $d_{I_y} | a_i$ for all $i \in I_x \setminus I_y$. \square

From this we deduce:

10.6. Lemma. *Assume Λ is of adjoint type. Then $\mathfrak{C}_{\mathcal{P}}$ has a finite filtration such that each factor is a constant sheaf supported on a face of \mathcal{P} .*

Proof. Let $\mathfrak{a} = \mathfrak{a}_0 \times \mathfrak{a}_1 \times \dots \times \mathfrak{a}_m$ and $\Phi = \Phi_1 \cup \dots \cup \Phi_m$ be the unique decomposition of (\mathfrak{a}, Φ) into a trivial part \mathfrak{a}_0 and irreducible parts $\mathfrak{a}_1, \dots, \mathfrak{a}_m$. Then $\mathfrak{C}_{\mathcal{P}} = \mathfrak{C}^{(1)} \oplus \dots \oplus \mathfrak{C}^{(m)}$ where $\mathfrak{C}^{(i)}$ is the pull-back of $\mathfrak{C}_{\mathcal{A}_i}$ to \mathcal{P} . Thus it suffices to show the assertion for $\mathfrak{C} := \mathfrak{C}^{(i)}$ for any i . We may also assume that Φ_i is infinite. Let $\alpha_1, \dots, \alpha_n \in \Phi_i$ be the simple roots.

For any prime power p^e let $\mathfrak{C}[p^e] \subseteq \mathfrak{C}_{\mathcal{P}}$ be the kernel of multiplication by p^e . The union $\mathfrak{C}[p^\infty]$ over all e is the p -primary component of $\mathfrak{C}_{\mathcal{P}}$. Since $\mathfrak{C}_{\mathcal{P}}$ is the direct sum of its primary components it suffices to show the assertion for $\mathfrak{C}[p^\infty]$. Now it follows from Lemma 10.5 that $\mathfrak{C}[p^e]/\mathfrak{C}[p^{e-1}]$ is a constant sheaf with stalks $\mathbb{Z}/p\mathbb{Z}$ which is supported in the face

$$\{x \in \mathcal{P} \mid \alpha_i(x) = 0 \text{ for all } i \text{ with } p^e \nmid a_i\} \quad \square \quad (10.23)$$

Since constant sheaves on contractible spaces have trivial cohomology, we get:

10.7. Corollary. *Assume Λ is of adjoint type. Then $H^i(\mathcal{P}, \mathfrak{C}_{\mathcal{P}}) = 0$ for all $i \geq 1$.*

Next we study the kernel of ε (eq. (10.5)). Observe that the constant sheaf $\mathbb{R}_{\mathcal{P}}$ is contained in this kernel. From this we get a homomorphism

$$\bar{\varepsilon} : \mathcal{C}_{\mathcal{P}}^W / \mathbb{R}_{\mathcal{P}} \rightarrow \mathfrak{L}_{\mathcal{P}, \Lambda}^{\Phi} \quad (10.24)$$

10.8. Lemma. *Let $\mathfrak{K}_{\mathcal{P}}$ be the kernel of $\bar{\varepsilon}$. Then its stalk in $x \in \mathcal{P}$ is equal to $\Lambda^{\vee} \cap (\mathbb{R}\overline{\Phi}_x)^{\vee}$.*

Proof. Let $x \in \mathcal{P}$ and $U \subseteq \mathcal{P}$ a small convex open neighborhood. A function f on U in the kernel of ε if and only if its gradient is in Γ^{\vee} . Continuity implies that ∇f must be in fact constant. This implies that f is an affine linear function with $\overline{f} \in \Gamma^{\vee}$. Moreover, f is a section of $\mathcal{C}_{\mathcal{P}}^W$ if and only if \overline{f} is W_x -invariant. This means, \overline{f} should be orthogonal to all $\overline{\alpha} \in \overline{\Phi}_x$. \square

10.9. Lemma. *Let Λ be of adjoint type and let $\alpha_1, \dots, \alpha_n$ be the simple roots of Φ . Let H_i be the hyperplane $\{\alpha_i = 0\}$. Then the sheaf $\mathfrak{K}_{\mathcal{P}}$ fits into an exact sequence*

$$0 \rightarrow \mathfrak{K}_{\mathcal{P}} \rightarrow \Lambda_{\mathcal{P}}^{\vee} \xrightarrow{\varrho} \bigoplus_{i=1}^n \mathbb{Z}_{H_i \cap \mathcal{P}} \xrightarrow{\psi} \mathfrak{C}_{\mathcal{P}} \rightarrow 0. \quad (10.25)$$

Proof. All sheaves are clearly restrictions of the corresponding sheaves on \mathcal{A} to \mathcal{P} . Thus we may assume that $\mathcal{P} = \mathcal{A}$. Then we may treat every factor of the root system Φ separately. Thus, we may assume that Φ is either finite or irreducible and infinite.

For $x \in \mathcal{P}$ we have to show that the stalk $\mathfrak{K}_x = \Lambda^{\vee} \cap (\mathbb{R}\overline{\Phi}_x)^{\vee}$ fits into an exact sequence

$$0 \rightarrow \mathfrak{K}_x \rightarrow \Lambda^{\vee} \xrightarrow{\varrho_x} \mathbb{Z}^{I_x} \xrightarrow{\psi_x} \mathfrak{C}_x \rightarrow 0. \quad (10.26)$$

First, we define ϱ_x as $\varrho_x(v) := (\langle \overline{\alpha}_i, v \rangle)_{i \in I_x} \in \mathbb{Z}^{I_x}$. Then \mathfrak{K}_x is clearly the kernel of ϱ_x .

If Φ is finite then the set of all $\overline{\alpha}_i$ with $i \in I_x$ is part of a dual basis of Λ^{\vee} . Thus ϱ_x is surjective. Thus (10.26) is exact since $\mathfrak{C}_x = 0$ in this case.

Now assume that Φ is irreducible and infinite. Then $\mathfrak{C}_x = \mathbb{Z}/d_{I_x}\mathbb{Z}$ and we define ψ_x as $\psi_x(y_i) := \sum_{i \in I_x} a_i y_i + d_{I_x}\mathbb{Z}$. Identifying $\overline{\alpha}$ with the hyperplane H as in the proof of Lemma 10.5 we have to show that

$$\mathbb{Z}^n \cap H \xrightarrow{\varrho_x} \mathbb{Z}^{I_x} \xrightarrow{\psi_x} \mathbb{Z}/d_I\mathbb{Z} \rightarrow 0 \quad (10.27)$$

is exact. Surjectivity follows again from $\gcd(d_I, d_{I'}) = 1$. Moreover, the kernel of ψ_x consists of all $(y_i)_{i \in I_x}$ which can be extended to an n -tuple $(y_i)_{i=1}^n \in \mathbb{Z}^n$ with $\sum_{i=1}^n a_i y_i = 0$. This is equivalent to $\sum_{i \in I_x} a_i y_i$ being divisible by d_I . \square

10.10. Lemma. *Assume Λ to be of adjoint type. Then the homomorphism*

$$H^0(\psi) : H^0(\mathcal{P}, \bigoplus_i \mathbb{Z}_{H_i \cap \mathcal{P}}) \rightarrow H^0(\mathcal{P}, \mathfrak{C}_{\mathcal{P}}) \quad (10.28)$$

is surjective.

Proof. Both sides decompose as direct sums according to the decomposition of Φ into factors. Thus we may assume that Φ is irreducible. Then there is nothing to prove if Φ is finite since then $\mathfrak{C}_{\mathcal{P}} = 0$. So assume that Φ is infinite.

Let p be a prime. Then it suffices to show $H^0(\psi)$ is surjective on p -primary components. For $e \geq 0$ let $\mathcal{F}_e \subseteq \mathcal{A}$ be the face of \mathcal{A} which is defined by the equations $\alpha_i = 0$ whenever $p^e \nmid a_i$. Then $\mathcal{A} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \dots$. Let e be maximal with $\mathcal{F}_e \cap \mathcal{P} \neq \emptyset$. Then the explicit description of \mathcal{C} (Lemma 10.5) and the convexity of \mathcal{P} yield that $H^0(\mathcal{P}, \mathcal{C})[p^{\infty}] = \mathbb{Z}/p^e\mathbb{Z}$.

We may assume that $e \geq 1$. Then there is a simple root α_{i_0} with $p \nmid a_{i_0}$. Since then $\alpha_{i_0}(x) = 0$ for all $x \in \mathcal{F}_e \cap \mathcal{P}$, the summand $\mathbb{Z}_{H_{i_0} \cap \mathcal{P}}$ contributes to a summand $\cong \mathbb{Z}$ in $H^0(\mathcal{P}, \bigoplus_i \mathbb{Z}_{H_i \cap \mathcal{P}})$. Moreover, the restriction of $H^0(\psi)$ to this summand is multiplication by a_{i_0} followed by reducing mod p^e . This yields the assertion. \square

10.11. Lemma. *Assume Λ is of adjoint type. Then $H^i(\mathcal{P}, \mathfrak{K}_{\mathcal{P}}) = 0$ for all $i \geq 2$.*

Proof. Let \mathfrak{T} be the kernel of ψ , yielding a short exact sequence

$$0 \rightarrow \mathfrak{T} \rightarrow \bigoplus_i \mathbb{Z}_{H_i \cap \mathcal{P}} \xrightarrow{\psi} \mathfrak{C}_{\mathcal{P}} \rightarrow 0. \quad (10.29)$$

Since $H_i \cap \mathcal{P}$ is convex, hence contractible, the higher cohomology of $\bigoplus_i \mathbb{Z}_{H_i \cap \mathcal{P}}$ vanishes. We already proved that $H^i(\mathcal{P}, \mathfrak{C}) = 0$ for all $i \geq 1$. Combined with the surjectivity of $H^0(\psi)$ this implies that $H^i(\mathcal{P}, \mathfrak{T}) = 0$ for all $i \geq 1$. Since also $H^i(\mathcal{P}, \Lambda^\vee) = 0$ for all $i \geq 1$, the short exact sequence

$$0 \rightarrow \mathfrak{K} \rightarrow \Lambda_{\mathcal{P}}^\vee \rightarrow \mathfrak{T} \rightarrow 0 \quad (10.30)$$

implies $H^i(\mathcal{P}, \mathfrak{K}_{\mathcal{P}}) = 0$ for all $i \geq 2$. \square

Proof of Theorem 10.1. By Lemma 10.2 we may assume that Λ is of adjoint type. Consider the short exact sequence

$$0 \rightarrow \mathbb{R}_{\mathcal{P}} \rightarrow \mathcal{C}_{\mathcal{P}}^W \rightarrow \mathcal{C}_{\mathcal{P}}^W / \mathbb{R}_{\mathcal{P}} \rightarrow 0. \quad (10.31)$$

Since $\mathcal{C}_{\mathcal{P}}^W$ is a soft sheaf and \mathcal{P} is contractible, the higher cohomology of all three sheaves vanishes. Let $\mathfrak{S} \subseteq \mathfrak{L}_{\mathcal{P}, \Lambda}^\Phi$ be the image of ε . Then we get a short exact sequence

$$0 \rightarrow \mathfrak{K}_{\mathcal{P}} \rightarrow \mathcal{C}_{\mathcal{P}}^W / \mathbb{R}_{\mathcal{P}} \rightarrow \mathfrak{S} \rightarrow 0. \quad (10.32)$$

Lemma 10.11 implies $H^i(\mathcal{P}, \mathfrak{S}) = 0$ for all $i \geq 1$. Finally, Corollary 10.7 and the short exact sequence

$$0 \rightarrow \mathfrak{S} \rightarrow \mathfrak{L}_{\mathcal{P}, \Lambda}^\Phi \rightarrow \mathfrak{C}_{\mathcal{P}} \rightarrow 0 \quad (10.33)$$

imply $H^i(\mathcal{P}, \mathfrak{L}_{\mathcal{P}, \Lambda}^\Phi) = 0$ for all $i \geq 1$. \square

11. Examples

We conclude this paper with a series of examples which is in no way comprehensive. It should be mentioned that Paulus has obtained in his thesis, [Pau], more complete results by classifying certain subclasses of multiplicity free manifolds. Some of the examples below are in fact due to him.

Given a spherical pair (\mathcal{P}, Λ) , how do we construct M with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}, \Lambda)$? In section 6, this problem was essentially reduced to results from [Kno11], which we wish to state more explicitly.

Choose any point $x \in \mathcal{P}$ and let $L \subseteq K$ be the centralizer of $\exp(x) \in K$. Then, by assumption, there is a smooth affine spherical $L_{\mathbb{C}}$ -variety X with $\Lambda_X^+ = C_x \mathcal{P} \cap \Lambda$ where Λ_X^+ is the set of highest weights occurring in the coordinate ring of X . We call X the *local model of (\mathcal{P}, Λ) in x* . By Losev's theorem, [Los09, Thm. 1.3], it is unique. Let $U \subseteq \mathcal{P}$ be a small open convex neighborhood of x . Then it essentially a result of Brion [Bri87] and Sjamaar [Sja98] that the cross-section $M_0 = \log M_U$ (see (5.6)) is isomorphic, as an L -Hamiltonian manifold, to an open subset $X_0 \subseteq X$ where the Hamiltonian structure on X is induced by a closed embedding into a unitary L -representation. It follows that

$m^{-1}(U)$ is isomorphic to the fiber product $K \times^L X_0$. The latter is an open subset of $K \times^L X$ which we therefore call the *big local model at x* .

Using Remark *ii*) after Theorem 5.4, there is even a canonical (maximal) choice for U , namely $\mathcal{P}_x := \mathcal{P} \setminus F$ where F is the union of all (closed) faces of \mathcal{P} which do not contain x . Since \mathcal{P} is locally polyhedral it is easily seen that $\mathcal{P}_x \subseteq \mathcal{P}$ is open and that x lies in the unique closed face of \mathcal{P}_x . Let $\mathcal{A}^\sigma \subseteq \mathcal{A}$ be the smallest face of the alcove \mathcal{A} containing x . Let $\mathcal{A}_\sigma \subseteq \mathcal{A}$ be as in (5.26). Then clearly $\mathcal{P} = \mathcal{P}_0 \subseteq \mathcal{A}_\sigma$. This entails that M_U is globally the “exponential” of Hamiltonian L -variety (M_0, m_0) with $(m_0)_+(M_0) = \mathcal{P}_x - x$.

The local model X always has the form $X = L_{\mathbb{C}} \times^{H_{\mathbb{C}}} V$ where $H \subseteq L$ is a closed subgroup and V is a (complex) representation of $H_{\mathbb{C}}$ (see [KVS06, Cor. 2.2]). The variety X contains a unique closed $L_{\mathbb{C}}$ -orbit namely $L_{\mathbb{C}}/H_{\mathbb{C}}$. This implies that the L -orbit L/H is isomorphic to the fiber $m^{-1}(x) \subseteq M$. Hence, the fiber of m_+ over x is K/H .

The above gives a way to check whether a given pair is spherical. In fact, by work of Pezzini-Van Steirteghem, [PVS15], this becomes a completely combinatorial problem. Constructing spherical pairs is more difficult, though. A frequently successful strategy for finding spherical pairs is to start with a local model X . To this end, Van Steirteghem and the author have compiled a comprehensive list of all smooth affine spherical varieties, see [KVS06]. The corresponding weight monoids Λ_X^+ are calculated in the forthcoming paper [KPVS]. This then yields the tangent cone $\mathcal{C}_x \mathcal{P}$ and the lattice Λ .

1. Disymmetric spaces

Nevertheless, we start with examples which are not obtained by the method detailed above. Recall that a symmetric space is a homogeneous space of the form K/K^τ where K^τ is the fixed point group of an involution τ of K . The product $K/K^\sigma \times K/K^\tau$ of two symmetric spaces together with the diagonal K -action will be called a *disymmetric space*. They form a very important class of multiplicity free manifolds:

11.1. Theorem. *Let K be a connected compact Lie group with two involutions σ, τ and let $M = K/K^\sigma \times K/K^\tau$ be the corresponding disymmetric space. Then M carries the structure of a multiplicity free quasi-Hamiltonian $K\sigma\tau$ -manifold such that the moment map is*

$$m : M \rightarrow K\sigma\tau : (aK^\sigma, bK^\tau) \mapsto a(\sigma a^{-1})(\sigma b)(\sigma^\tau b^{-1}) \quad (11.1)$$

Proof. According to [AMM98, 3.1] any conjugacy class is a quasi-Hamiltonian manifold with the inclusion into the group being the moment map. Apply this to the group $\mathbb{Z}\sigma \ltimes K$. Then the conjugacy class of σ can be identified with K/K^σ . Hence, the symmetric space K/K^σ is a quasi-Hamiltonian $K\sigma$ -manifold with moment map

$$K/K^\sigma \rightarrow K\sigma : aK^\sigma \mapsto a(\sigma a^{-1}) \quad (11.2)$$

Analogously, the space K/K^τ is a quasi-Hamiltonian $K\tau$ -manifold. Hence the fusion product $M = K/K^\sigma \otimes K/K^\tau$ (see [AMM98] or proof of Proposition 5.2) is quasi-Hamiltonian with twist $\sigma\tau$. The moment map of M is just the product of the two moment maps of the factors which amounts to formula (11.1).

So far, we have not even used that σ and τ are involutions. This only comes in to show that M is multiplicity free. From (11.1) we get that the derivative of m at $z := (1K^\sigma, 1K^\tau) \in M$ is

$$Dm : \mathfrak{k}/\mathfrak{k}^\sigma \times \mathfrak{k}/\mathfrak{k}^\tau \rightarrow \mathfrak{k} : (\xi, \eta) \mapsto \xi - {}^\sigma\xi + {}^\sigma\eta - {}^{\sigma\tau}\eta. \quad (11.3)$$

Now let $\mathfrak{p}_\sigma, \mathfrak{p}_\tau \subseteq \mathfrak{k}$ be the (-1) -eigenspaces of σ, τ , respectively. Then $\mathfrak{k}/\mathfrak{k}^\sigma \times \mathfrak{k}/\mathfrak{k}^\tau \cong \mathfrak{p}_\sigma \times \mathfrak{p}_\tau$ and (11.3) becomes

$$Dm : \mathfrak{p}_\sigma \times \mathfrak{p}_\tau \rightarrow \mathfrak{k} : (\xi, \eta) \mapsto 2\xi + 2^\sigma \eta = 2^\sigma (\eta - \xi). \quad (11.4)$$

Thus, $\ker Dm$ equals $\mathfrak{q} := \mathfrak{p}_\sigma \cap \mathfrak{p}_\tau$ which is embedded diagonally into $\mathfrak{p}_\sigma \times \mathfrak{p}_\tau$. For $\xi \in \mathfrak{q}$ and $t \in \mathbb{R}$ we have

$$e^{t\xi} \cdot m(z) = e^{t\xi} \cdot 1 \cdot e^{-\sigma\tau(t\xi)} = 1. \quad (11.5)$$

Hence $\mathfrak{q}m(z) = 0$, i.e., \mathfrak{q} is parallel to the fiber $M_z := m^{-1}(m(z))$. On the other hand $\mathfrak{q}z = \mathfrak{q} = \ker Dm$ which implies that $K_{m(z)}$ acts transitively on M_z . There was nothing particular about the point z since every point $(aK^\sigma, bK^\tau) \in M$ has the form of z after replacing σ, τ by their conjugates with a, b , respectively. This shows that the symplectic reductions $M_z/K_{m(z)}$ are discrete for all $z \in M$, i.e., M is multiplicity free. \square

Of special importance is the case when σ and τ represent the same element in $\text{Out}(K)$. Then $\sigma\tau$ is inner and we get:

11.2. Corollary. *Let K, σ, τ and M as above. Assume that there is an element $u \in K$ with $\sigma\tau = \text{Ad}(u)$. Then M carries the structure of an (untwisted) multiplicity free quasi-Hamiltonian K -manifold such that the moment map is*

$$m : M \rightarrow K : (aK^\sigma, bK^\tau) \mapsto a(\sigma a^{-1}) u (\tau b)b^{-1} \quad (11.6)$$

Proof. Follows from (see (11.1))

$$\begin{aligned} m(aK^\sigma, bK^\tau) &= a\sigma a^{-1}b\tau b^{-1} = \\ &= a \cdot \sigma a^{-1} \sigma \cdot \sigma\tau \cdot \tau b\tau \cdot b^{-1} = a(\sigma a)^{-1} u (\tau b)b^{-1} \end{aligned} \quad (11.7)$$

\square

The orbit structure of disymmetric spaces has been thoroughly investigated by Matsuki in a series of papers [Mat95, Mat97, Mat02]. In particular, he showed that the orbit space M/K can be identified with a polytope which is of course our \mathcal{P}_M . Its shape is controlled by a Weyl group which is closely related to our W_M . In fact, it is possible to derive our invariants \mathcal{P}_M , Λ_M , and Φ_M from Matsuki's calculations. Since the results do not really fit into the present paper, details will appear elsewhere.

Nevertheless, to show the idea, we will give one instructive example without proofs, namely where $K = SU(2n)$ with $n \geq 2$ and σ and τ are the involutions defining the subgroups $K^\sigma = SO(2n)$ and $K^\tau = Sp(2n)$, respectively. It is well known that

$$\mathcal{A} = \{(x_1, \dots, x_{2n}) \mid x_1 \geq \dots \geq x_{2n} \geq x_1 - 1 \text{ and } x_1 + \dots + x_{2n} = 0\} \quad (11.8)$$

is the fundamental alcove for $SU(2n)$. The simple roots are

$$\alpha_1 := x_1 - x_2, \dots, \alpha_{2n-1} = x_{2n-1} - x_{2n}, \alpha_{2n} = x_{2n} - x_1 + 1 = \alpha_0 \quad (11.9)$$

It follows from Matsuki's calculations that the momentum image of M is

$$\mathcal{P}_M = \left\{ (y_1 + \frac{1}{4}, \dots, y_n + \frac{1}{4}, y_1 - \frac{1}{4}, \dots, y_n - \frac{1}{4}) \mid \begin{array}{l} y_1 \geq \dots \geq y_n \geq y_1 - \frac{1}{2} \\ y_1 + \dots + y_n = 0 \end{array} \right\} \quad (11.10)$$

which implies

$$\mathfrak{a}_M = \{x_1 - x_{n+1} = \dots = x_n - x_{2n} = \frac{1}{2}, x_1 + \dots + x_{2n} = 0\} \quad (11.11)$$

The simple roots of Φ_M are

$$\sigma_1 = \alpha_1 + \alpha_{n+1}, \dots, \sigma_n = \alpha_n + \alpha_{2n}. \quad (11.12)$$

which shows that Φ_M is an affine root system of type $A_{n-1}^{(1)}$. The character group Λ_M of M is just the weight lattice of $\bar{\Phi}_M$ in $\bar{\mathfrak{a}}_M$. The point $x \in \mathcal{P}_M$ with coordinates $y_1 = \dots = y_n = 0$ is a vertex of \mathcal{P}_M . Since all simple roots of K except for α_0 and α_n vanish in x the centralizer of $\exp(x)$ in K is the Levi subgroup $L = S(U(n) \times U(n))$. Its complexification is $L_{\mathbb{C}} = S(GL(n, \mathbb{C}) \times GL(n, \mathbb{C}))$. So the local model X in x must have the weight monoid $\Lambda_X^+ = \mathcal{C}_x \mathcal{P}_M \cap \Lambda_M$. Let $H = GL(n, \mathbb{C})$ be embedded into $L_{\mathbb{C}}$ via $g \mapsto (g, (g^t)^{-1})$. Then one checks that $L_{\mathbb{C}}/H$ has indeed the desired weight monoid and therefore equals X . Observe that $X = SL(n, \mathbb{C})$ with $L_{\mathbb{C}}$ -action $(g_1, g_2)g = g_1 g g_2^t$. The big local model in x is

$$Z := SU(2n) \times^L X. \quad (11.13)$$

The other vertices are all conjugate under the center of K and therefore have isomorphic local models. The upshot is that $M = SU(2n)/SO(2n) \times SU(2n)/Sp(2n)$ can be obtained by gluing n copies of Z but in a non-holomorphic way.

Another instance of a disymmetric manifold is the double $D(K_0)$ of a connected compact Lie group (see also the proof of Proposition 5.2). Put $K := K_0 \times K_0$ and let τ be the switching automorphism. Then K^τ is the diagonal in K and $K/K^\tau = K_0$ is a τ -twisted conjugacy class. Therefore, $D(K_0) = K_0 \times K_0$ is a multiplicity free quasi-Hamiltonian K -manifold with diagonal action

$$(x, y) * (a, b) = (xay^{-1}, xby^{-1}) \quad (11.14)$$

and moment map

$$m(a, b) = (ab^{-1}, a^{-1}b). \quad (11.15)$$

After the coordinate change

$$D(K_0) \rightarrow D(K_0) : (a, b) \mapsto (a, b^{-1}) \quad (11.16)$$

this is precisely the double in the sense of [AMM98, §3.2]. Let \mathcal{A}_0 and P_0 be the alcove and the weight lattice of K_0 . Then $\mathcal{A} = \mathcal{A}_0 \times \mathcal{A}_0$ and $P = P_0 \oplus P_0$ are alcove and weight lattice of K . Let, moreover, $\delta(\chi) := -w_0(\chi)$ be the opposition endomorphism of \mathfrak{a}_0 where w_0 is longest element of the Weyl group of K_0 . Then

$$(\mathcal{P}_{D(K_0)}, \Lambda_{D(K_0)}) = (\text{id} \times \delta)(\mathcal{A}_0, P_0) \subseteq (\mathcal{A}, P) \quad (11.17)$$

are the data of $D(K_0)$.

Remark. Already the case $M = D(SU(2))$ exhibits a new phenomenon namely that the classification of multiplicity free manifolds depends on the choice of the invariant scalar product on \mathfrak{k} . To see this, observe that the alcove \mathcal{A} for $K = SU(2) \times SU(2)$ is a rectangle whose side lengths depend on the metric. Let \mathcal{P} be a diagonal of \mathcal{A} . Then in order to be spherical, \mathcal{P} has to be parallel to the sum $\alpha + \alpha'$ of the simple roots. This holds if and only if \mathcal{A} is a square, i.e., the metrics on both factors of K are the same.

2. Groups of rank 1

Let $K = SU(2)$. Then \mathcal{A} is an interval and $\mathcal{P} \subseteq \mathcal{A}$ is a subinterval. If $\mathcal{P}_M \neq \mathcal{A}$ then the discussion in Remark ii) after Theorem 5.4 shows that M is the exponential of a Hamiltonian manifold. Quasi-Hamiltonian manifolds which are not of this form will be called *genuine*. So a genuine multiplicity free quasi-Hamiltonian $SU(2)$ -manifold

necessarily has $\mathcal{P}_M = \mathcal{A}$. The possible local models in the end points are the $SL(2, \mathbb{C})$ -varieties \mathbb{C}^2 , $SL(2, \mathbb{C})/\mathbb{C}^*$ and $SL(2, \mathbb{C})/N(\mathbb{C}^*)$. Accordingly, there are 3 different genuine multiplicity free quasi-Hamiltonian $SU(2)$ -manifolds namely:

- $\Lambda_M = P$ ($\cong \mathbb{Z}$, the weight lattice of $SU(2)$). Here M is obtained by gluing two copies of \mathbb{C}^2 and is therefore diffeomorphic to the 4-sphere S^4 . This example has been found by by Alekseev-Meinrenken-Woodward [AMW02] under the name “spinning 4-sphere”.
- $\Lambda_M = 2P$. Then M is the disymmetric manifold $M = S^2 \times S^2$.
- $\Lambda_M = 4P$. In this case, M is the quotient of the previous one by the switching involution. Hence $M \cong \mathbf{P}^2(\mathbb{C})$.

There is another affine root system of rank 1 namely $A_2(2)$. It belongs to $K = SU(3)$ and an outer automorphism τ of K , e.g. complex conjugation. The alcove \mathcal{A} is an interval and the two roots α_0, α_1 satisfy $\bar{\alpha}_0 = -2\bar{\alpha}_1$. The weight lattice is $P = \mathbb{Z}\bar{\alpha}_1$. The centralizers corresponding to the end points are $SU(2)$ and $SO(3)$, respectively. Let $\mathcal{P}_M = \mathcal{A}$. Then a discussion as above yields two cases

- $\Lambda_M = P$: In this case, the local models are \mathbb{C}^2 and $SO(3, \mathbb{C})/SO(2, \mathbb{C})$.
- $\Lambda_M = 2P$. In this case, the local models are $SL(2, \mathbb{C})/SO(2, \mathbb{C})$ and $SO(3, \mathbb{C})/O(2, \mathbb{C})$.

Note that $\Lambda_M = 4P$ does not work since $\Lambda_X^+ = \mathbb{Z}_{\geq 0}(4\bar{\alpha}_1)$ is not the weight monoid of any smooth $SO(3, \mathbb{C})$ -variety.

3. Manifolds of rank 1

The spinning 4-sphere has been generalized in [HJS06] by Hurtubise-Jeffrey-Sjamaar to that of a spinning $2n$ -sphere. In our terms it can be constructed as follows: let $K = SU(n)$. Then the alcove \mathcal{A} has n vertices namely $x_0 = 0$ and the fundamental weights $x_i = \omega_i$, $i = 1, \dots, n-1$. Let \mathcal{P} be the edge joining x_0 and x_1 . Let $\Lambda = \mathbb{Z}\omega_1$. Then (\mathcal{P}, Λ) with $\Lambda = \mathbb{Z}\omega_1$ is a spherical pair. Indeed, the weights of the smooth affine spherical $SL(n, \mathbb{C})$ -variety $X = \mathbb{C}^n$ form the monoid $\mathbb{Z}_{\geq 0}\omega_1$. This shows that it is a local model at the vertex x_0 . The situation in x_1 is similar: the centralizer is still $L = K = SU(n)$ but the simple root system is different, namely $\alpha_2, \alpha_3, \dots, \alpha_{n-1}, \alpha_n = \alpha_0$. The last fundamental weight with respect to this system is $-\omega_1$. Therefore the monoid $C_{x_1}\mathcal{P} \cap \Lambda = \mathbb{Z}_{\geq 0}(-\omega_1)$ has a model, as well, namely again \mathbb{C}^n . Glued together this yields the spinning $2n$ -sphere.

Eshmatov, [Esh09], has found an analogue of the spinning $2n$ -sphere for the symplectic group. More precisely, he showed that $M = \mathbf{P}(\mathbb{H}^n)$ carries the structure of a multiplicity free quasi-Hamiltonian $Sp(2n)$ -manifold. Using our theory, this structure can be obtained as follows. Let $K = Sp(2n)$ and let $\varepsilon_1, \dots, \varepsilon_n$ be the standard basis of the Cartan subalgebra \mathfrak{t} . Let \mathcal{P} be the line segment joining the origin $x_0 = 0$ with $x_1 = \frac{1}{2}\varepsilon_1$. This is an edge of the fundamental alcove \mathcal{A} . Put $\Lambda := \mathbb{Z}\varepsilon_1$. Then

$$X_1 = \mathbb{C}^{2n} \tag{11.18}$$

is an (even big) local model in x_0 . The other endpoint x_1 behaves differently, though. In this case the simple roots of L are $\alpha_0, \alpha_2, \alpha_3, \dots, \alpha_n$ which yields $L = Sp(2) \times Sp(2n-2)$. Moreover $-\omega_1$ is now the fundamental weight of the first factor of L . Thus, the smooth affine spherical variety with character group $\mathbb{Z}_{\geq 0}(-\omega_1)$ is simply \mathbb{C}^2 with the second factor

of L acting trivially. As a big local model at x_1 we obtain

$$X_2 = Sp(2n) \times_{Sp(2) \times Sp(2n-2)} \mathbb{C}^2. \quad (11.19)$$

Now Eshmatov's space is obtained by gluing X_1 and X_2 .

This example has been further generalized by Paulus. We keep $K = Sp(2n)$. Then the vertices of \mathcal{A} are $x_k := \sum_{i=1}^k \varepsilon_i$ for $k = 0, \dots, n$. Fix k with $k > 0$ and let \mathcal{P}_k be the line segment joining x_{k-1} and x_k . Let moreover $\Lambda_k := \mathbb{Z}\varepsilon_k$. Then one shows as above that $(\mathcal{P}_k, \Lambda_k)$ is spherical and it is even possible to identify the corresponding manifold:

11.3. Theorem. *Let n, k be integers with $1 \leq k \leq n$. Then there is a multiplicity free quasi-Hamiltonian $Sp(2n)$ -manifold structure on the quaternionic Grassmannian $M = \text{Gr}_k(\mathbb{H}^{n+1})$ with $(\mathcal{P}_M, \Lambda_M) = (\mathcal{P}_k, \Lambda_k)$.*

Proof. The big local models at x_{k-1} and x_k , respectively are the spaces

$$X_1 := Sp(2n) \times^{H_{k-1}} \mathbb{C}^{2n-2k+2} \text{ and } X_2 := Sp(2n) \times^{H_k} \mathbb{C}^{2k} \quad (11.20)$$

where $H_k := Sp(2k) \times Sp(2n-2k) \subseteq Sp(2n)$ and they glue to a multiplicity free quasi-Hamiltonian manifold M . Now recall that $Sp(2n)$ can also be interpreted as the unitary group of \mathbb{H}^n . Then H_k is the isotropy group of $\mathbb{H}^k \subseteq \mathbb{H}^n$. Therefore X_2 can be identified with the universal bundle $\widetilde{\text{Gr}}_k(\mathbb{H}^n)$ over the quaternionic Grassmannian $\text{Gr}_k(\mathbb{H}^n)$. Similarly, X_1 is isomorphic to $\widetilde{\text{Gr}}_{n-k+1}(\mathbb{H}^n)$. Now consider the space $\mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H}$ where K acts on the first factor. Let $e := (0, 1)$ be the fixed point. Each element of $\widetilde{\text{Gr}}_k(\mathbb{H}^n)$ can be interpreted as a pair (L, v) with $L \in \text{Gr}_k(\mathbb{H}^n)$ and $v \in L$. Let $\Gamma_{L,v} \subseteq \mathbb{H}^n \oplus \mathbb{H}$ be the graph of the map $L \rightarrow \mathbb{H} : u \mapsto \langle u, v \rangle$. Then the map $(L, v) \mapsto \Gamma_{L,v}$ identifies $X_2 = \widetilde{\text{Gr}}_k(\mathbb{H}^n)$ with the open subset of all $\tilde{L} \in \text{Gr}_k(\mathbb{H}^{n+1})$ with $e \notin \tilde{L}$. Similarly, X_1 be identified with the set of all $\tilde{L} \in \text{Gr}_k(\mathbb{H}^{n+1})$ with $e \notin \tilde{L}^\perp$. So $\text{Gr}_k(\mathbb{H}^{n+1})$ is also obtained by gluing X_1 and X_2 . It is easy to see that there is only one K -equivariant way to do that, so $M \cong \text{Gr}_k(\mathbb{H}^{n+1})$ (see e.g. [GWZ08]). \square

Remark. The complex Grassmannians $\text{Gr}_k(\mathbb{C}^{n+1})$ are multiplicity free quasi-Hamiltonian, as well, namely for $K = SU(n)$. But they are not genuine, i.e., they are “exponentials” of (ordinary) Hamiltonian manifolds.

5. Surjective moment maps

It is interesting to look at multiplicity free quasi-Hamiltonian manifolds M which are in a sense as big as possible. That means, first of all, that \mathcal{P}_M is the entire alcove. This is clearly equivalent to the moment map being surjective. There are quite a few of them, most of which are disymmetric. Therefore we also demand that Λ_M be as big as possible, i.e., equals the weight lattice P of $\overline{\Phi}$. This is equivalent to the K -action on M being free.

11.4. Theorem. *Let (K, τ) be one of the following three cases:*

$$(SU(n), \text{id}), \quad (Sp(2n), \text{id}), \quad (SU(2n+1), k \mapsto \overline{k}) \quad (11.21)$$

(the last τ is complex conjugation). Then (\mathcal{A}, P) is spherical, i.e., there is a unique multiplicity free quasi-Hamiltonian $K\tau$ -manifold M whose moment map is surjective and such that K acts freely on M .

Proof. It suffices to find a local model in each of the vertices x of \mathcal{A} . For that, each case will be treated separately.

$(K, \tau) = (SU(n), \text{id})$: We start with $x = 0 \in \mathcal{P} = \mathcal{A}$. Then $L = K$ and $C_x \mathcal{A}$ is the dominant Weyl chamber. Therefore, we have to show that there is a smooth affine $SL(n, \mathbb{C})$ -variety X_n such that $\mathcal{O}(X) = \bigoplus_{\chi} L(\chi)$ where χ runs through all dominant weights. Such a variety does in general not exist for an arbitrary reductive group but it does for $SL(n, \mathbb{C})$ namely

$$X_n := \begin{cases} SL(n, \mathbb{C}) \overset{Sp(n, \mathbb{C})}{\times} \mathbb{C}^n & \text{if } n \text{ is even,} \\ SL(n, \mathbb{C})/Sp(n-1, \mathbb{C}) & \text{if } n \text{ is odd.} \end{cases} \quad (11.22)$$

Thus (\mathcal{A}, Λ) is spherical in $x = 0$. But then it is also spherical in all other vertices since they differ only in a translation by an element of the center.

$(K, \tau) = (Sp(2n), \text{id})$: A model in $x = 0$ is

$$Y_n := \begin{cases} Sp(2n, \mathbb{C}) \overset{Sp(n, \mathbb{C}) \times Sp(n, \mathbb{C})}{\times} \mathbb{C}^n & \text{if } n \text{ is even,} \\ Sp(2n, \mathbb{C}) \overset{Sp(n-1, \mathbb{C}) \times Sp(n+1, \mathbb{C})}{\times} \mathbb{C}^{n+1} & \text{if } n \text{ is odd.} \end{cases} \quad (11.23)$$

In general, \mathcal{A} has $n+1$ vertices $x_0 = 0, x_1, \dots, x_n$ which are enumerated in such a way that $\alpha_i(x_i) \neq 0$. Then the centralizer in L in x_i is $L = Sp(2i) \times Sp(2n-2i)$. Since $\Lambda = \mathbb{Z}^n = \mathbb{Z}^i \oplus \mathbb{Z}^{n-i}$ splits accordingly, the varieties

$$Y_{i, n-i} := Sp(2n) \overset{Sp(2i) \times Sp(2n-2i)}{\times} (Y_i \times Y_{n-i}) \quad (11.24)$$

are the big local models in x_i .

$(K, \tau) = (SU(2n+1), \tau)$ with τ an outer automorphism: Here, the Dynkin diagram of (K, τ) is of type $A_{2n}^{(2)}$ (see (9.5)). In this case, \mathcal{A} has $n+1$ vertices x_0, \dots, x_n such that the centralizer of x_i is $L = Sp(2i) \times SO(2n+1-2i)$. It is well-known that the coordinate ring of

$$Z_n := SO(2n+1, \mathbb{C})/GL(n, \mathbb{C}) \quad (11.25)$$

contains all irreducible $SO(2n+1, \mathbb{C})$ -modules exactly once. So

$$Z_{i, n-i} := SU(2n+1) \overset{Sp(2i) \times SO(2n+1-2i)}{\times} (Y_i \times Z_{n-i}) \quad (11.26)$$

is a big local model in x_i . \square

Remarks. 1. Using the spherical roots of the local models it is easy to determine the root system Φ_M in each case:

Untwisted $SU(n)$: The simple affine roots of K are

$$\alpha_0 = 1 + x_n - x_1, \alpha_1 = x_1 - x_2, \dots, \alpha_{n-1} = x_{n-1} - x_n. \quad (11.27)$$

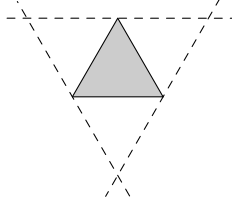
The spherical roots of X_n are $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \dots, \alpha_{n-2} + \alpha_{n-1}$. For n odd, see [BP15]. The even case is handled in [KPVS] or, in this case, in [Lun07]. Therefore, the simple roots of Φ_M are

$$1 + x_n - x_2, x_1 - x_3, x_2 - x_4, \dots, x_{n-2} - x_n, 1 + x_{n-1} - x_1. \quad (11.28)$$

Hence

$$\Phi_M \cong \begin{cases} A_{\frac{n}{2}-1}^{(1)} \times A_{\frac{n}{2}-1}^{(1)} & n \text{ even,} \\ A_{n-1}^{(1)} & n \text{ odd.} \end{cases} \quad (11.29)$$

Observe that in the odd case the root systems of K and M are isomorphic but they are not the same. For example, for $n = 3$, i.e., $K = SU(3)$, one gets this picture:



(11.30)

where the gray triangle denotes $\mathcal{P} = \mathcal{A}$ and the axes of the simple reflections of Φ_M are marked by dashed lines. There is also something to be observed in the even case: here all roots of Φ_M are perpendicular to the vector $\delta = (1, -1, \dots, 1, -1) \in \bar{\mathfrak{a}}$. This means that the automorphism group of M contains the 1-dimensional torus $\exp(\mathbb{R}\delta) \subseteq A$, i.e., the $SU(n)$ -action on M extends to an $U(1) \times SU(n)$ -action.

$Sp(2n)$: The simple affine roots of K are

$$\alpha_0 = 1 - 2x_1, \alpha_1 = x_1 - x_2, \dots, \alpha_{n-1} = x_{n-1} - x_n, \alpha_n = 2x_n. \quad (11.31)$$

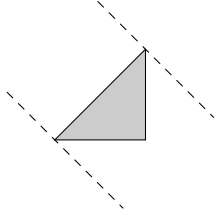
The spherical roots of Y_n are $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \dots, \alpha_{n-1} + \alpha_n$ (see [KPVS, Lum07]). Therefore, the simple roots of Φ_M are

$$1 - x_1 - x_2, x_1 - x_3, x_2 - x_4, \dots, x_{n-2} - x_n, x_{n-1} + x_n. \quad (11.32)$$

Hence

$$\Phi_M \cong A_{n-1}^{(1)} \quad (11.33)$$

For $K = Sp(4)$ one gets



(11.34)

Since there is again a W_M -invariant vector in $\bar{\mathfrak{a}}$, namely $\delta = (1, -1, 1, -1, \dots)$ the $Sp(n)$ -action on M extends to an $U(1) \times Sp(n)$ -action.

Twisted $SU(2n + 1)$: The simple affine roots of K, τ are

$$\alpha_0 = \frac{1}{2} - 2x_1, \alpha_1 = x_1 - x_2, \dots, \alpha_{n-1} = x_{n-1} - x_n, \alpha_n = x_n. \quad (11.35)$$

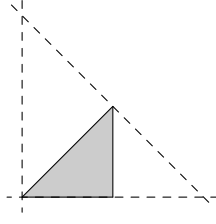
According [BP15], the spherical roots of Z_n are $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \dots, \alpha_{n-1} + \alpha_n, \alpha_n$ but here one has to be careful since the normalization of spherical roots is different from ours: in [BP15] all roots are always primitive in the weight lattice while our roots are always “as long as possible”. See [VS13] for further details. This amounts to changing the last root α_n to $2\alpha_n$. From this, we get the simple roots of Φ_M as

$$\frac{1}{2} - x_1 - x_2, x_1 - x_3, x_2 - x_4, \dots, x_{n-2} - x_n, x_{n-1}, 2x_n \quad (11.36)$$

Hence

$$\Phi_M \cong A_{2n}^{(2)}. \quad (11.37)$$

The fact that Φ_M has the same type as the root system of (K, τ) seems to be coincidental. For $K = SU(5)$ one gets



(11.38)

2. Paulus, [Pau], worked out a complete list of multiplicity free quasi-Hamiltonian manifolds (possibly twisted) with surjective moment map.

6. Inscribed triangles

For the last example, we were toying around with triangles inscribed in an triangular alcove. Here are some examples of spherical pairs (\mathcal{P}, Λ) :

K	$SU(3)$	$SU(3)$	$Sp(4)$	$Sp(4)$	G_2
$\mathcal{P} \subseteq \mathcal{A}$					
Λ	P or R	R	R	R	R

(11.39)

We make no claim of completeness. In particular, we considered only untwisted groups. The letters P and R denote the weight and the root lattice of K , respectively. At each vertex, the complexified centralizer L is isogenous to $SL(2, \mathbb{C}) \times \mathbb{C}^*$. Then one can show that the local models are either of the form $X = SL(2, \mathbb{C})/\mu_n$ in case \mathcal{P} touches \mathcal{A} in form of a reflection and $X = SL(2, \mathbb{C}) \times^{\mathbb{C}^*} \mathbb{C}$ if a right angle is involved.

Remark. As communicated by Eckhart Meinrenken, the first case has also been found by Chris Woodward (unpublished).

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